NASA TECHNICAL NOTE

NASA TN D-4033

c.1

NASA TN D-4033

ESTIMATION IN MIXED FREQUENCY DISTRIBUTIONS

by

A. Clifford Cohen, Jr.

University of Georgia
Institute of Statistics
Athens, Ga.

and

L. W. Falls

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1967

NASA TN D-4033

ESTIMATION IN MIXED FREQUENCY DISTRIBUTIONS

By A. Clifford Cohen, Jr.

University of Georgia
Institute of Statistics
Athens, Ga.

and

L. W. Falls

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

TABLE OF CONTENTS (Cont'd)

LIST OF ILLUSTRATIONS

LIST OF TABLES

ESTIMATION IN MIXED FREQUENCY DISTRIBUTIONS

SUMMARY

In the analysis of atmospheric data, distributions which are asymmetrical or multimodal are often encountered. These distributions are usually composed of two or more distinct homogeneous subpopulations and are designated as "mixed" or "compound" frequency distributions.

The problem of estimating parameters in a mixed distribution is of considerably greater complexity than that of estimation in a single unimodal distribution. Not only is it necessary to estimate the parameters of each component, but the proportionality factors, which express the proportion or percentage of each component in the mixture, must also be estimated. In the case of a mixture consisting of K components each having two parameters, there are 3K-1 parameters to be estimated in the resultant distribution.

Karl Pearson solved such a problem as early as 1894 for a compound normal population, using the method of moments. The problem was investigated later, first by Charlier, and then in a joint effort by Charlier and Wicksell, who greatly simplified the theory.

The papers presented in this document are concerned with estimation in compound distributions and are the results of an investigation performed by the Institute of Statistics, University of Georgia, Athens, Georgia, as a part of NASA contract NAS 8-11175 with the Terrestrial Environment Branch, Aerospace Environment Division, Aero-Astrodynamics Laboratory, NASA - George C. Marshall Space Flight Center, Huntsville, Alabama. Dr. A. C. Cohen, Jr., is the principal investigator and the NASA contract monitors are Mr. O. E. Smith and Mr. L. W. Falls.

The methods and procedures presented are practical and are applicable to experiments in which the data exhibit qualities that require a mixed distribution as the statistical model.

ESTIMATION IN A MIXTURE OF A POISSON WITH A NEGATIVE BINOMIAL DISTRIBUTION

by A. Clifford Cohen, Jr.

Introduction

Many of the distributions encountered in the analysis of atmospheric data are the result of mixing two or more separate component distributions. Such distributions are therefore of particular interest to aerospace scientists. Mixtures of two Poisson distributions, mixtures of two exponential distributions and mixtures of two normal distributions have been considered in previous memoranda [1,2,3]. This paper is concerned with estimation of the parameters in a distribution consisting of a Poisson component mixed with a negative binomial component.

The Probability Function

The density function of a mixed distribution with components $f_1(x)$ and $f_2(x)$ combined in the proportions α and $(1 - \alpha)$ respectively may be written as

$$f(x) = \alpha f_1(x) + (1 - \alpha) f_2(x) \ . \tag{1}$$

Let $f_1(x)$ be the density function of the Poisson component, which we write as

$$f_1(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad ; \quad x = 0, 1, 2 \ldots \ . \tag{2}$$

Let $f_2(x)$ be the density function of the negative binomial component, which may be written as

$$f(x) = \frac{\Gamma(x + k)}{x! \ \Gamma(k)} p^k (1 - p)^x \quad ; \quad x = 0, 1, 2 \ldots , \tag{3}$$

as

$$f(x) = \frac{\Gamma(x+k)}{x!\ \Gamma(k)} \frac{q^x}{(1+q)^{k+x}} \ ; \quad x = 0,\ 1,\ 2 \ldots , \tag{4}$$

or as

$$f(x) = \frac{\Gamma(x+k)}{x!\ \Gamma(k)} \left(1 + \frac{m}{k}\right)^{-k} \left(\frac{m}{m+k}\right)^{x} \ ; \quad x = 0,\ 1,\ 2 \ldots , \tag{5}$$

where $q = (1 - p)/p$, $m = qk$; $m, k, q > 0$; $0 < p < 1$.

Estimation of Parameters

When $f_2(x)$ assumes the form given in equation (5), the first four factorial moments and the zero probability of the mixed distribution are

$$\left.\begin{aligned}
\mu_{[1]} &= \alpha\lambda + (1 - \alpha)m, \\
\mu_{[2]} &= \alpha\lambda^2 + (1 - \alpha)(k + 1)m^2/k , \\
\mu_{[3]} &= \alpha\lambda^3 + (1 - \alpha)(k + 1)(k + 2)m^3/k^2 , \\
\mu_{[4]} &= \alpha\lambda^4 + (1 - \alpha)(k + 1)(k + 2)(k + 3)m^4/k^3 , \\
f(0) &= P = \alpha e^{-\lambda} + (1 - \alpha)(1 + m/k)^{-k} .
\end{aligned}\right\} \tag{6}$$

On equating the above moments to corresponding sample moments and on setting $P = n_0/n$ for a random sample of size n where n_0 is the number of zero observations in the sample, we obtain a system of equations, any four of which might be solved simultaneously for estimates of the parameters α, λ, m and k. Here, we consider estimates based on (a) the first four moments, and (b) on the first three moments and selected frequencies.

Equate $\mu_{[1]}$, $\mu_{[2]}$ and $\mu_{[3]}$ to $\bar{x}$, $\nu_{[2]}$ and $\nu_{[3]}$ respectively; the first three moments of (6) become

$$\left.\begin{aligned} \bar{x} &= \alpha\lambda + (1-\alpha)m, \\ \nu_{[2]} &= \alpha\lambda^2 + (1-\alpha)(k+1)m^2/k, \\ \nu_{[3]} &= \alpha\lambda^3 + (1-\alpha)(k+1)(k+2)m^3/k^2, \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} \bar{x} &= \sum_{x=0}^{R} x\, n_x/n, \\ \nu_{[j]} &= \sum_{x=0}^{R} x(x-1)\ldots(x-j+1)n_x/n, \end{aligned}\right\} \tag{8}$$

in which n_x denotes the number of sample observations for which the random variable $X = x$, and R denotes the largest observed value of X.

Let us assume that λ is known. We then solve the first equation of (7) for m and thereby obtain

$$m = \frac{\bar{x} - \alpha\lambda}{1-\alpha} . \tag{9}$$

On substituting this value into the last two equations of (7), and simplifying, we obtain

$$\left(\frac{k+1}{k}\right) = (1-\alpha)\,\frac{(\nu_{[2]} - \alpha\lambda^2)}{(\bar{x} - \alpha\lambda)^2} = \psi_1(\alpha \mid \lambda)\,,$$

$$\left(\frac{k+2}{k}\right) = \frac{(1-\alpha)\,(\nu_{[3]} - \alpha\lambda^3)}{(\bar{x} - \alpha\lambda)(\nu_{[2]} - \alpha\lambda^2)} = \psi_2(\alpha \mid \lambda)\,. \qquad (10)$$

With λ thus assumed to be known, we eliminate k between the two equations of (10) to obtain a quadratic equation in α which we write as

$$\alpha^2 - B\alpha + C = 0\,, \qquad (11)$$

where

$$B = \frac{(\bar{x}\nu_{[3]} - 2\nu_{[2]}^2) + \lambda(\nu_{[3]} + 2\bar{x}\nu_{[2]}) - \lambda^2(4\nu_{[2]} - \bar{x}^2) + \lambda^3\bar{x}}{\lambda\nu_{[3]} - 3\lambda^2\nu_{[2]} + 3\lambda^3\bar{x} - \lambda^4}\,,$$

$$C = \frac{\bar{x}\nu_{[3]} + (\bar{x}^2 - 2\nu_{[2]})\,\nu_{[2]}}{\lambda\nu_{[3]} - 3\lambda^2\nu_{[2]} + 3\lambda^3\bar{x} - \lambda^4}\,. \qquad (12)$$

The required estimate α^* must be a positive root of (11) such that $0 < \alpha^* < 1$ and such that corresponding estimates m^* and k^* are both positive. For some combinations of sample data, such an estimate may not exist and for some samples, both roots of (11) may satisfy the criteria set forth above for acceptable estimates of α. In the former case we can conclude only that either the sample data fail to conform to the model specified by equation (1) or the sample is too small. When the solution of equation (11) produces two acceptable estimates of α, we might choose the one which produces the closest agreement between the expected and the observed fourth moment or between the expected and observed number of observations in some specified class. For example, the zero class or perhaps the modal class might be chosen.

With α^* determined, we estimate m from (9) as

$$m^* = \frac{\overline{x} - \lambda\alpha^*}{1 - \alpha^*} , \tag{13}$$

and k from the first equation of (10) as

$$k^* = [\psi_1(\alpha^* \mid \lambda) - 1]^{-1} , \tag{14}$$

or as

$$k^* = \left[\frac{\nu_{[2]} - \alpha^* \lambda^2}{(1 - \alpha^*)(m^*)^2} - 1\right]^{-1} . \tag{15}$$

The asterisks employed here serve to distinguish estimators or estimates from the parameters being estimated. When desired, estimates of p and q as employed in (3) and (4) may be expressed as

$$q^* = m^*/k^* , \quad p^* = 1/(1 + q^*) . \tag{16}$$

<u>Estimates Based on First Four Moments</u>. The foregoing results are based on the assumption that λ is known. If indeed this assumption were true, our task would now be complete. In the more general case, where λ too must be estimated from the sample data, we might begin by assuming a value $\lambda_{(1)}$ as a first approximation and subsequently compute first approximations $\alpha_{(1)}$, $m_{(1)}$ and $k_{(1)}$ using (11), (13) and (14), or (15). On substituting these values into the fourth equation of (6), we calculate a first approximation to the fourth factorial moment, ${\mu_{[4]}}_{(1)}$ for comparison with the corresponding sample moment $\nu_{[4]}$. Once we have found two values $\lambda_{(i)}$ and $\lambda_{(i+1)}$ in a sufficiently narrow interval such that $\nu_{[4]}$ is in the interval $(\mu_{[4](i)}, \mu_{[4](i+1)})$, we can interpolate linearly as indicated in Table I to determine final estimates α^*, λ^*, m^*, k^* for which the first four sample moments are in agreement with the first four distribution moments.

TABLE I. LINEAR INTERPOLATION FOR FINAL ESTIMATES

λ	α	m	k	$\mu_{[4]}$
$\lambda_{(i)}$	$\alpha_{(i)}$	$m_{(i)}$	$k_{(i)}$	$\mu_{[4](i)}$
$\underline{\lambda^*}$	$\underline{\alpha^*}$	$\underline{m^*}$	$\underline{k^*}$	$\underline{\nu_{[4]}}$
$\lambda_{(i+1)}$	$\alpha_{(i+1)}$	$m_{(i+1)}$	$k_{(i+1)}$	$\mu_{[4](i+1)}$

Estimates Based on First Three Moments Plus Frequency in a Selected Class. Because of larger sampling errors that are inherent in higher moments, it might sometimes be desirable to exchange the fourth equation of (6) which involved the fourth moment for an equation setting the observed proportion of observations equal to the expected proportion in a selected class. For example, the zero class might be chosen in which case we would employ the fifth equation of (6) rather than the fourth. In general, however, we might wish to employ the modal class. We begin with a first approximation $\lambda_{(1)}$ and subsequently determine $\alpha_{(1)}$, $m_{(1)}$ and $k_{(1)}$ from the first three moments using equations (11), (13) and (14) or (15), as before. We then compute $f_{(1)}(x)$ for the selected value of x using equation (1) and compare the value thus obtained with the observed proportion n_x/n. Subsequently, we determine second approximations $\lambda_{(2)}$, $\alpha_{(2)}$, $m_{(2)}$ and $k_{(2)}$. With these values substituted in (1), we compute $f_{(2)}(x)$. As soon as two values $\lambda_{(i)}$ and $\lambda_{(i+1)}$ in a sufficiently narrow interval have been found such that n_x/n is in the interval $[f_{(i)}(x), f_{(i+1)}(x)]$, we interpolate linearly as indicated in Table II for the final estimates λ^{**}, α^{**}, m^{**}, and k^{**}.

The double asterisks distinguish estimates based on the first three moments and a selected density from those based on the first four moments (designated with single asterisks) and from parameters being estimated (without asterisks).

TABLE II. LINEAR INTERPOLATIONS FOR FINAL ESTIMATES

λ	α	m	k	f(x)
$\lambda_{(i)}$	$\alpha_{(i)}$	$m_{(i)}$	$k_{(i)}$	$f_{(i)}(x)$
$\underline{\lambda^{**}}$	$\underline{\alpha^{**}}$	$\underline{m^{**}}$	$\underline{k^{**}}$	$\underline{n_x/n}$
$\lambda_{(i+1)}$	$\alpha_{(i+1)}$	$m_{(i+1)}$	$k_{(i+1)}$	$f_{(i+1)}(x)$

In both of the estimating procedures presented here, a first approximation to λ in the vicinity of $\overline{x}$ should usually be satisfactory.

An Illustrative Example

To illustrate the practical application of estimating procedures developed here, we construct a sample consisting of 1000 observations from a mixed Poisson and negative binomial distribution for which $\alpha = 0.4$, $\lambda = 3$, $m = 4.5$ and $k = 3$. Data for this sample are shown in Table III.

For this sample, $n = 1000$, $\overline{x} = 3.915$, $\nu_{[2]} = 20.034$, $\nu_{[3]} = 134.946$ and $\nu_{[4]} = 1,138.536$. On substituting these values into (12), we have

$$B = \frac{-274.408722 + 291.81222\lambda - 64.808775\lambda^2 + 3.915\lambda^3}{134.946\lambda - 60.102\lambda^2 + 11.745\lambda^3 - \lambda^4},$$

$$C = \frac{32.65690365}{134.946\lambda - 60.102\lambda^2 + 11.745\lambda^3 - \lambda^4}.$$

TABLE III. SAMPLE FROM MIXED POISSON AND NEGATIVE BINOMIAL DISTRIBUTION

x	n_x	x	n_x
0	58	10	16
1	129	11	11
2	172	12	8
3	172	13	5
4	142	14	4
5	103	15	2
6	71	16	2
7	47	17	1
8	32	18	1
9	22	19	1
		20	1

As a first approximation, we let $\lambda_{(1)} = 3.2$ and when this value is substituted in the above expressions for B and C, equation (11) becomes

$$\alpha^2 - 1.28688692\,\alpha + 0.33881630 = 0 .$$

Solving with the aid of the quadratic formula, we find

$$\alpha_{(1)} = \frac{1}{2}\,[1.28688692 \pm \sqrt{0.30081276}\,] ,$$

from which

$$\alpha_{(1)} = 0.36921 \quad \text{or} \quad 0.91768 .$$

When the smaller of these values is substituted into (13) and (15), we find

$$m_{(1)} = \frac{3.915 - (0.36921)(3.2)}{0.63079} = 4.3335 ,$$

$$k_{(1)} = [1.372078942 - 1]^{-1} = 2.6876 .$$

When the larger root of the quadratic equation is substituted into (13) and (15), we obtain a negative value for k. Since, however, k must be positive, the larger root is rejected and the values given above as based on the smaller root are accepted as first approximations to m and k and the smaller root itself is accepted as our first approximation to α. Using the values thus calculated, we employ the fourth equation of (6) to calculate as a first approximation to the fourth factorial moment

$$\mu_{[4]} = 1165.32 ,$$

which is to be compared with the sample value $\nu_{[4]} = 1138.536$.

For a second approximation, we choose $\lambda_{(2)} = 2.8$, and following the same procedure as with the first approximations, we find $\alpha_{(2)} = 0.42645$, $m_{(2)} = 4.7440$, $k_{(2)} = 3.4127$ and $\mu_{[4](2)} = 1145.72$. We ultimately try a third approximation $\alpha_{(3)} = 2.7$ for which $\mu_{[4](3)} = 1138.27$. The results of these different approximations are summarized in Table IV and the tables following.

In solving equation (11) for various values of λ as shown above, the smaller root was chosen as the appropriate approximation to the estimate of α in each case as the larger root, even when otherwise acceptable, resulted in a greater disparity between observed and expected number of observations in the zero class.

TABLE IV. FIRST APPROXIMATIONS

λ	Estimating Equations
2.6	$\alpha^2 - 1.09214560\alpha + 0.31012438 = 0$
2.7	$\alpha^2 - 1.13280360\alpha + 0.31327564 = 0$
2.8	$\alpha^2 - 1.16985786\alpha + 0.31702728 = 0$
3.0	$\alpha^2 - 1.23410203\alpha + 0.32645477 = 0$
3.2	$\alpha^2 - 1.28688692\alpha + 0.33881630 = 0$

TABLE V. SECOND APPROXIMATION

λ	α	m	k	q	p	$\mu_{[4]}$
2.6	Imaginary					
2.7	0.47960	5.0347	3.9421	1.2772	0.4391	1138.27
2.8	0.42645	4.7440	3.4127	1.3901	0.4184	1145.72
3.0	0.38403	4.4855	2.9614	1.5146	0.3977	1155.96
3.2	0.36922	4.3335	2.6876	1.6124	0.3828	1165.32

Additional computations were made with $\lambda = 2.6$ and $\lambda = 3$. With $\lambda = 2.6$, equation (11) has no real roots and thus this value must be rejected. Approximations corresponding to $\lambda = 3$ are included in Table V. For final estimates based on the first four moments, we interpolate between approximations corresponding to $\lambda = 2.7$ and $\lambda = 2.8$ as shown in Table VI.

TABLE VI. FINAL ESTIMATES BASED ON FIRST FOUR MOMENTS

λ	α	m	k	$\mu_{[4]}$
2.8	0.42645	4.7440	3.4127	1145.72
<u>2.704</u>	<u>0.47777</u>	<u>5.0242</u>	<u>3.9229</u>	<u>1138.54</u>
2.7	0.47960	5.0347	3.9421	1138.27

Accordingly, final moment estimates are $\lambda^* = 2.704$, $\alpha^* = 0.4778$, $m^* = 5.024$, $k^* = 3.923$, $q^* = 1.280$, $p^* = 0.439$, which are to be compared with the actual distribution parameters, $\lambda = 3$, $\alpha = 0.4$, $m = 4.5$, $k = 3$, $q = 1.5$, $p = 0.4$.

Somewhat improved estimates are obtained when the frequency in the one class rather than the fourth factorial moment is utilized as the basis for our fourth estimating equation. Estimates, thus based on the first three moments and the frequency in the one class are determined by interpolation as indicated in Table VII, where for a given value of λ, f(1) is calculated using the probability

TABLE VII. INTERPOLATION DETERMINING ESTIMATES BASED ON THE FIRST THREE MOMENTS

λ	α	m	k	f(1)
3.0	0.38403	4.4855	2.9614	0.12896
<u>2.996</u>	<u>0.38495</u>	<u>4.4911</u>	<u>2.9712</u>	<u>0.12900</u>
2.8	0.42645	4.7440	3.4127	0.13080

function (1) with values for α, m, and k based on the first three moments, determined as previously described. Final estimates based on the first three moments and the frequency of ones are thus $\lambda^{**} = 2.996$, $\alpha^{**} = 0.385$, $m^{**} = 4.49$, $k^{**} = 2.97$ and from (16) $q^{**} = 1.51$ and $p^{**} = 0.40$.

Essentially these same estimates are obtained, when our calculations are based on frequencies in the zero, two, three, or four classes or even on a combination of these classes rather than on the one class as has been done here. The improvement achieved by equating observed frequencies to expected frequencies rather than equating the observed fourth moment to the expected fourth moment apparently reflects the lack of stability in the higher sample moments. The fact that very small changes in α result in quite large variations in the higher moments is also involved. For comparison all estimates calculated are listed in Table VIII alongside the true parameter values.

TABLE VIII. ALL ESTIMATES ALONGSIDE TRUE PARAMETER VALUES

Basis	λ	α	m	k	q	p
Parameter Values	3	0.4	4.5	3	1.5	0.4
First Four Moments	2.704	0.478	5.02	3.92	1.28	0.44
First Three Moments and Freq. of Ones	2.996	0.385	4.49	2.97	1.51	0.40

ESTIMATION IN MIXTURES OF TWO NORMAL DISTRIBUTIONS

by A. Clifford Cohen, Jr.

Summary

This paper is concerned primarily with the method of moments in dissecting a mixture of two normal distributions. In the general case, with two means, two standard deviations, and a proportionality factor to be estimated, the first five sample moments are required, and it becomes necessary to find a particular solution of a ninth degree polynomial equation that was originally derived by Karl Pearson [4]. A procedure which circumvents solution of the nonic equation and thereby considerably reduces the total computational effort otherwise required, is presented. Estimates obtained in the simpler special case in which the two standard deviations are assumed to be equal are employed as first approximations in an iterative method for simultaneously solving the basic system of moment equations applicable in the more general case in which the two standard deviations are unequal. Conditional maximum likelihood and conditional minimum chi-square estimation subject to having the first four sample moments equated to corresponding population moments, are also considered. An illustrative example is included.

Research was sponsored by the Aero-Astrodynamics Laboratory of the Marshall Space Flight Center, National Aeronautics and Space Administration; Contract NAS8-11175.

Introduction

Distributions which result from the mixing of two or more component distributions are designated as "compound" or "mixed" distributions. They may be further described by designating distribution types of the individual components. Such distributions arise in a wide variety of practical situations ranging from distributions of wind velocities to distributions of physical dimensions of various mass produced items. Compound normal distributions were studied as early as 1894 by Karl Pearson [4] and later by Charlier [5] and by Charlier and Wicksell [6]. More recently, compound Poisson and compound exponential distributions have been studied by Rider [7] and by Cohen [1, 2, 8, 9]. Compound binomial distributions have been studied by Blischke [10].

This paper is concerned with estimation of the parameters θ_1, σ_1, θ_2, σ_2, and α of the compound normal distribution with density

$$f(x) = \alpha f_1(x) + (1 - \alpha) f_2(x) \ , \tag{1}$$

where

$$\left.\begin{aligned} f_1(x) &= (2\pi\sigma_1^2)^{-1/2} \exp-[(x - \theta_1)/\sigma_1]^2/2, \\ f_2(x) &= (2\pi\sigma_2^2)^{-1/2} \exp-[(x - \theta_2)/\sigma_2]^2/2 \ . \end{aligned}\right\} \tag{2}$$

Karl Pearson [4] derived estimators for the parameters of this distribution by equating sample moments to corresponding population (theoretical) moments. The evaluation of his estimators involved the solution of a ninth degree polynomial equation. Before the advent of modern electronic computers, this was considered a rather formidable obstacle to the application of his results. Charlier and Wicksell [6] succeeded in considerably simplifying Pearson's results. The present effort represents an attempt to simplify these estimators further in order that they might be more readily available for use in appropriate practical applications. A procedure is presented whereby the direct solution of Pearson's nonic equation can be circumvented through an iterative process which involves solving a cubic equation for a unique negative root. In addition to considering the most general case in which all five parameters of (1) must be estimated from the data available in a given sample, some of the more important special cases in which one or more of the parameters are known in advance of making sample observations are also examined.

Estimation in the General Case

Except for a few changes in notation, the presentation of this section is essentially that of Charlier and Wicksell [6].

The kth moment of $f(x)$ taken about the origin may be written as

$$\mu_k' = \alpha \int_{-\infty}^{\infty} x^k f_1(x)\,dx + (1 - \alpha) \int_{-\infty}^{\infty} x^k f_2(x)\,dx \ , \tag{3}$$

where $f_1(x)$ and $f_2(x)$ are density functions of the two component distributions as given in equations (2). The k<u>th</u> central moment becomes

$$\mu_k = \alpha \int_{-\infty}^{\infty} (x - \mu_1')^k f_1(x)\,dx + (1 - \alpha) \int_{-\infty}^{\infty} (x - \mu_1')^k f_2(x)\,dx \,. \tag{4}$$

If we let

$$m_1 = \theta_1 - \mu_1' \quad \text{and} \quad m_2 = \theta_2 - \mu_1' \;, \tag{5}$$

the first non-central and the first five central moments of (1) follow as

$$\left.\begin{aligned}
\mu_1' &= \alpha\theta_1 + (1 - \alpha)\theta_2 \;, \\
\mu_1 &= \alpha m_1 + (1 - \alpha) m_2 = 0 \;, \\
\mu_2 &= \alpha(\sigma_1^2 + m_1^2) + (1 - \alpha)(\sigma_2^2 + m_2^2) \;, \\
\mu_3 &= \alpha m_1(3\sigma_1^2 + m_1^2) + (1 - \alpha) m_2(3\sigma_2^2 + m_2^2) \;, \\
\mu_4 &= \alpha[3\sigma_1^4 + 6m_1^2\sigma_1^2 + m_1^4] + (1 - \alpha)[3\sigma_2^4 + 6m_2^2\sigma_2^2 + m_2^4] \;, \\
\mu_5 &= \alpha m_1[15\sigma_1^4 + 10m_1^2\sigma_1^2 + m_1^4] + (1 - \alpha) m_2[15\sigma_2^4 + 10m_2^2\sigma_2^2 + m_2^4] \;.
\end{aligned}\right\} \tag{6}$$

Without any loss of generality let us suppose that $\theta_1 \leq \theta_2$. Accordingly $\theta_1 \leq \mu_1' \leq \theta_2$, and $m_1 \leq 0 \leq m_2$.

Upon equating population moments to corresponding sample moments, it follows from (6) that

$$\alpha[\theta_1 - \overline{x}] + (1 - \alpha)[\theta_2 - \overline{x}] = 0 \;, \tag{7}$$

and further that

$$\left.\begin{aligned}
&\alpha m_1 + (1-\alpha) m_2 &&= 0, \\
&\alpha[\sigma_1^2 + m_1^2 - \nu_2] + (1-\alpha)[\sigma_2^2 + m_2^2 - \nu_2] &&= 0, \\
&\alpha[3\sigma_1^2 m_1 + m_1^3 - \nu_3] + (1-\alpha)[3\sigma_2^2 m_2 + m_2^3 - \nu_3] &&= 0, \\
&\alpha[3\sigma_1^4 + 6m_1^2\sigma_1^2 + m_1^4 - \nu_4] \\
&\qquad + (1-\alpha)[3\sigma_2^4 + 6m_2^2\sigma_2^2 + m_2^4 - \nu_4] &&= 0, \\
&\alpha[15\sigma_1^4 m_1 + 10\sigma_1^2 m_1^3 + m_1^5 - \nu_5] \\
&\qquad + (1-\alpha)[15\sigma_2^4 m_2 + 10\sigma_2^2 m_2^3 + m_2^5 - \nu_5] &&= 0,
\end{aligned}\right\} \quad (8)$$

where $\bar{x}$ is the sample mean and ν_i, $(i = 2, 3, 4, \ldots)$ is the ith central moment of the sample. Equations (8) accordingly constitute a system of five equations to be solved simultaneously for estimates of the five parameters α, m_1, σ_1, m_2, σ_2.

We eliminate α between the first and subsequent equations of (8) in turn and thereby reduce this system to the following four equations in the four unknowns σ_1, m_1, σ_2, m_2:

$$\left.\begin{aligned}
&\frac{\sigma_1^2 + m_1^2 - \nu_2}{m_1} = \frac{\sigma_2^2 + m_2^2 - \nu_2}{m_2} = \beta, \\
&\frac{3\sigma_1^2 m_1 + m_1^3 - \nu_3}{m_1} = \frac{3\sigma_2^2 m_2 + m_2^3 - \nu_3}{m_2},
\end{aligned}\right\} \quad (9)$$

(Equation (9) concluded on following page.)

$$\left.\begin{aligned}\frac{3\sigma_1^4 + 6m_1^2\sigma_1^2 + m_1^4 - \nu_4}{m_1} &= \frac{3\sigma_2^4 + 6m_2^2\sigma_2^2 + m_2^4 - \nu_4}{m_2} ,\\ \frac{15\sigma_1^4 m_1 + 10\sigma_1^2 m_1^3 + m_1^5 - \nu_5}{m_1} &= \frac{15\sigma_2^4 m_2 + 10\sigma_2^2 m_2^3 + m_2^5 - \nu_5}{m_2} .\end{aligned}\right\} \quad \begin{array}{c}(9)\\ \text{(Con-}\\ \text{cluded)}\end{array}$$

In view of the presence of m_1 and m_2 in the denominators of (9), we recognize that these equations and those subsequently derived from them are not valid in the symmetric case to be dealt with later in which $m_1 = m_2 = 0$. In all subsequent results presented in this section it is understood that $m_1 < 0$ and $m_2 > 0$.

With the introduction of β in the first equation of (9) it follows that

$$\left.\begin{aligned}\sigma_1^2 &= m_1\beta + \nu_2 - m_1^2 ,\\ \sigma_2^2 &= m_2\beta + \nu_2 - m_2^2 .\end{aligned}\right\} \qquad (10)$$

On replacing σ_1^2 and σ_2^2 in the second, third and fourth equations of (9) with the values given in (10) above, it follows after considerable algebraic manipulation that

$$\left.\begin{aligned}&m_1m_2[3\beta - 2(m_1 + m_2)] = -\nu_3 ,\\ &m_1m_2[3\beta^2 - 2(m_1^2 + m_1m_2 + m_2^2)] = -k_4 ,\\ &m_1m_2[15(m_1 + m_2)\beta^2 - 20(m_1^2 + m_1m_2 + m_2^2)\beta\\ &\qquad + 6(m_1 + m_2)(m_1^2 + m_2^2)] = -k_5 ,\end{aligned}\right\} \qquad (11)$$

where k_4 and k_5 are respectively the fourth and fifth order sample cumulants or semi-invariants; i. e.,

$$\left.\begin{aligned} k_4 &= \nu_4 - 3\nu_2^2 , \\ k_5 &= \nu_5 - 10\nu_2\nu_3 . \end{aligned}\right\} \tag{12}$$

When referring to population (theoretical) cumulants, we employ the Greek kappa thus:

$$\kappa_4 = \mu_4 - 3\mu_2^2 , \qquad \kappa_5 = \mu_5 - 10\mu_2\mu_3 .$$

Equations (11) accordingly constitute a system of three equations in the three unknowns, β, m_1 and m_2.

In order to further simplify the system of equations (11), let

$$r = m_1 + m_2, \quad \text{and} \quad v = m_1 m_2 . \tag{13}$$

When these transformations are introduced into equations (11), the system becomes

$$\left.\begin{aligned} & 3\beta v - 2rv = -\nu_3 , \\ & 3\beta^2 v - 2v(r^2 - v) = -k_4 , \\ & 15vr\beta^2 - 20v(r^2 - v)\beta + 6vr(r^2 - 2v) = -k_5 . \end{aligned}\right\} \tag{14}$$

On making the further transformation

$$w = rv , \tag{15}$$

the system of equations (14) becomes

$$
\left.
\begin{aligned}
&3\beta v - 2w = -\nu_3 \ , \\
&3\beta^2 v^2 - 2w^2 + 2v^3 = -k_4 v \ , \\
&15v^2 w\beta^2 - 20vw^2\beta + 20v^4\beta + 6w^3 - 12v^3 w = -k_5 v^2 \ .
\end{aligned}
\right\} \tag{16}
$$

We now eliminate β between the first and the second and between the first and the third equations of (16), and our system becomes

$$
\left.
\begin{aligned}
&2w^2 - 6v^3 - 3vk_4 + 4w\nu_3 - \nu_3^2 = 0 \ , \\
&5w(2w - \nu_3)^2 - 20w^2(2w - \nu_3) \\
&\qquad + 20v^3(2w - \nu_3) + 18w^3 - 36wv^3 = -3k_5 v^2 \ .
\end{aligned}
\right\} \tag{17}
$$

By introducing the further transformation

$$z = w + \nu_3 \ ,$$

the two equations of (17) become

$$
\left.
\begin{aligned}
&2z^2 = 6v^3 + 3k_4 v + 3\nu_3^2 \ , \\
&2z^2(z - 3\nu_3) + z(\nu_3^2 - 4v^3) + 3\nu_3^3 + 24\nu_3 v^3 = 3k_5 v^2 \ .
\end{aligned}
\right\} \tag{19}
$$

When the expression for $2z^2$ from the first of the above equations is substituted into the second of those equations, we obtain

$$z = \frac{-6\nu_3 v^3 + 3k_5 v^2 + 9\nu_3 k_4 + 6\nu_3^3}{2v^3 + 3k_4 v + 4\nu_3^2} \quad , \tag{20}$$

and when this value is reinserted into the first equation of (19), we obtain a polynomial equation of the ninth degree in v, which for convenience we write as follows:

$$\left.\begin{aligned} & a_9 v^9 + a_8 v^8 + a_7 v^7 + a_6 v^6 + a_5 v^5 + a_4 v^4 + a_3 v^3 \\ & \qquad + a_2 v^2 + a_1 v + a_0 = 0 \ , \end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned} a_9 &= 8 \ , \\ a_8 &= 0 \ , \\ a_7 &= 28 k_4 \ , \\ a_6 &= 12\nu_3^2 \ , \\ a_5 &= 30 k_4^2 + 24 k_5 \nu_3 \ , \\ a_4 &= 148 k_4 \nu_3^2 - 6 k_5^2 \ , \\ a_3 &= 96\nu_3^4 - 36\nu_3 k_4 k_5 + 9 k_4^3 \ , \\ a_2 &= -(21\nu_3^2 k_4^2 + 24\nu_3^3 k_5) \ , \\ a_1 &= -32\nu_3^4 k_4 \ , \\ a_0 &= -8\nu_3^6 \ , \end{aligned}\right\} \tag{22}$$

This is the well known nonic which was first given in 1894 by Pearson [4].

Since here $m_1 < 0$ and $m_2 > 0$, then $v = m_1 m_2 < 0$, and the required estimate v^* is to be found as a negative real root of (21). Throughout this paper asterisks (*) are employed to distinguish estimates from the parameters being estimated. Prior to the advent of electronic computers, the task of solving this equation would have been considered a formidable assignment. Today, however, modern computers are available to perform the otherwise burdensome calculations involved. The ninth degree polynomial can be evaluated for any desired value of v in the vicinity of v^* either by straightforward substitution or by synthetic division. Standard iterative procedures will quickly lead to the required value of v^*. Once v^* is determined with the desired degree of accuracy, the estimate w^* follows from (20) and (18) as

$$w^* = \frac{-8\nu_3 v^{*3} + 3k_5 v^{*2} + 6\nu_3 k_4 v^* + 2\nu_3^3}{2v^{*3} + 3k_4 v^* + 4\nu_3^2} .$$

From (15), we have $r^* = w^*/v^*$, and accordingly

$$r^* = \frac{-8\nu_3 v^{*3} + 3k_5 v^{*2} + 6\nu_3 k_4 v^* + 2\nu_3^3}{v^*(2v^{*3} + 3k_4 v^* + 4\nu_3^2)} . \tag{23}$$

It follows from the defining relations (13), that estimates of m_1 and m_2 are the roots of the quadratic equation

$$Y^2 - r^* Y + v^* = 0 . \tag{24}$$

Thus, we have

$$\left.\begin{aligned} m_1^* &= \frac{1}{2}\,[r^* - \sqrt{r^{*2} - 4v^*}\,], \\ m_2^* &= \frac{1}{2}\,[r^* + \sqrt{r^{*2} - 4v^*}\,]. \end{aligned}\right\} \tag{25}$$

Using the first equation of (14), we estimate β as

$$\beta^* = \frac{1}{3}\,[2r^* - \nu_3/v^*] , \tag{26}$$

and from (10), we have

$$\left.\begin{aligned} \sigma_1^{*2} &= m_1^*\beta^* + \nu_2 - m_1^{*2} , \\ \sigma_2^{*2} &= m_2^*\beta^* + \nu_2 - m_2^{*2} . \end{aligned}\right\} \tag{27}$$

Finally, from (5) and from the second equation of (6), we have

$$\left.\begin{aligned} \theta_1^* &= m_1^* + \bar{x} , \\ \theta_2^* &= m_2^* + \bar{x} , \\ \alpha^* &= m_2^*/(m_2^* - m_1^*) . \end{aligned}\right\} \tag{28}$$

Attention is again drawn to the fact that the preceding results are valid only if $\theta_1 \neq \theta_2$. The symmetric case in which $\theta_1 = \theta_2$ and thus $m_1 = m_2 = r = v = w = 0$ is treated separately in the section entitled "Estimation in the Symmetric Cases."

Unfortunately, for some combinations of sample data the nonic equation (21) may have more than one negative root and accordingly we must choose between two or more sets of estimates. This lack of uniqueness bothered Pearson, and he suggested choosing the set of estimates which resulted in the closest agreement between the sixth central moment of the sample and the corresponding moment of the "fitted" compound curve.

On Circumventing the Nonic Estimating Equation. In the event that r is known, we need only consider the first two equations of (14), and when β is eliminated between them, we have the following cubic equation in v:

$$6v^3 - 2r^2v^2 + (3k_4 - 4r\nu_3)v + \nu_3^2 = 0 . \tag{29}$$

Using Descartes' rule of signs, we find that this equation has one negative root plus either two positive or a pair of imaginary roots. The negative root is the required estimate v^*.

Using this value for v^* and the known value for r, the required estimates follow from (25), (26), (27) and (28) as before.

Even though r is not known a priori, we might assume a value and employ the foregoing results to determine approximations to the required estimates which in turn can be substituted into the final equation of (6), to approximate the fifth central moment, μ_5.

Let $r_{(i)}$ designate the ith approximation to r^* and let $\mu_{5(i)}$ designate the ith approximation to μ_5^* based on $r_{(i)}$. It should be relatively easy to find approximations $r_{(i)}$ and $r_{(i+1)}$ such that the sample moment ν_5 is in the interval $(\mu_{5(i)}, \mu_{5(i+1)})$. As shown in the following section, a satisfactory initial approximation to r can usually be found by assuming $\sigma_1 = \sigma_2$ and solving the appropriate estimating equation for this special case. Once the interval between $r_{(i)}$ and $r_{(i+1)}$ has been narrowed sufficiently, the required estimate r^* can be obtained by simple linear interpolation as indicated below.

r	μ_5
$r_{(i)}$	$\mu_{5(i)}$
$\underline{r^*}$	$\underline{\nu_5}$
$r_{(i+1)}$	$\mu_{5(i+1)}$

With r specified, the well known method of Horner which utilizes synthetic division procedures is quite effective in solving (29) for v. Any standard iterative method, however, might be employed. A "trial and error" procedure based on linear interpolation with direct substitution in (29), though perhaps not very economical of computational effort, will generally give satisfactory results.

Various special cases in which one or more of the parameters are known or in some way restricted are sometimes of interest. With fewer parameters to be estimated, the number of sample moments involved is correspondingly reduced and the estimating equations are accordingly simpler and easier to solve. Some of the more important special cases are considered in the next two sections.

Conditional Maximum Likelihood and Conditional Minimum Chi-Square Estimation. In order to eliminate the effect of sampling errors resulting from direct use of the fifth order moment, let us consider a conditional maximum likelihood procedure. The first four sample moments are equated to corresponding population moments, and subject to this condition, r is determined so as to maximize

$$L'(r) = \prod_{i=1}^{n} f(x_i | \bar{x}, \nu_2, \nu_3, \nu_4) .$$

Since derivatives of $L'(r)$ are somewhat unwieldy, the value of r which maximizes $L'(r)$ can conveniently be determined in most practical applications by actually calculating $L'(r)$ for several values of r in the vicinity of its maximum and employing either finite difference or graphical techniques.

Grouped Data. In the case of grouped data, the conditional likelihood function might be expressed as

$$L'(r) = \prod_{i=1}^{k} p_i^{n_i} ,$$

where k is the number of groups or classes into which the sample has been divided, n_i is the number of sample observations in the ith class, x_i is the upper boundary of the ith class, and

$$p_i = \int_{x_{i-1}}^{x_i} f(x | \bar{x}, \nu_2, \nu_3, \nu_4) dx .$$

In practice it is sometimes more convenient to minimize chi-square than to maximize $L'(r)$. Kendall [11 (Vol. II, pp. 55-56)] has shown that with grouped data, the two methods are equivalent to the order $n^{-1/2}$. We are therefore free to choose the method requiring the least computational effort. In the method of minimum chi-square, we seek the value of r which results in the minimum value for

$$\chi^2(r|\overline{x}, \nu_2, \nu_3, \nu_4) = \sum_{i=1}^{k} \frac{(n_i - e_i)^2}{e_i} ,$$

where $e_i = np_i$ is the expected number of observations in the $i\underline{th}$ class subject to the restriction that the first four sample moments are equated to corresponding population moments. As previously indicated, n_i is the number of sample observations in the $i\underline{th}$ class.

In practice, chi-square can be calculated for several values of r in the vicinity of its minimum and the desired value of r can be determined either graphically or by employing finite difference techniques.

Unrestricted maximum likelihood estimates would, of course, be preferable to any moment estimates, but with five parameters to be estimated, the unrestricted maximum likelihood estimating equations become quite intractable. The conditional maximum likelihood procedures suggested here are believed to represent a feasible compromise between the need for estimating equations that are easy to solve and the need for reliable estimating procedures.

Estimation When $\sigma_1 = \sigma_2 = \sigma$

Here, we need only estimate the four parameters, θ_1, θ_2, α and σ where as in the general case $\theta_1 < \theta_2$. Accordingly only the first four moments and or cumulants are required. Charlier and Wicksell [6] and Rao [12] among others have previously considered this special case. With $\sigma_1 = \sigma_2 = \sigma$, equations (10) of the general five-parameter case now become

$$\sigma^2 = m_1\beta + \nu_2 - m_1^2 = m_2\beta + \nu_2 - m_2^2 . \tag{30}$$

From this it follows that

$$\left.\begin{aligned} \beta &= m_1 + m_2 = r , \\ \sigma^2 &= m_1 m_2 + \nu_2 = v + \nu_2 , \end{aligned}\right\} \tag{31}$$

where r and v are as defined in (13).

The first two equations of (14) which are applicable here now become

$$3rv - 2rv = -\nu_3 \ ,$$

$$3r^2v - 2v(r^2 - v) = -k_4 \ ,$$

and subsequently

$$\left.\begin{aligned} rv &= -\nu_3 \ , \\ r^2v + 2v^2 &= -k_4 \ . \end{aligned}\right\} \tag{32}$$

From the first of the above equations

$$r = -\nu_3/v \ . \tag{33}$$

When this value is substituted into the second equation of (32), and both sides are multiplied by v, the applicable estimating equation becomes

$$2v^3 + k_4v + \nu_3^2 = 0 \ . \tag{34}$$

From Descartes' rule of signs it follows that unless $\nu_3^2 = 0$, equation (34) has a single negative root, which is the required estimate v^* regardless of whether k_4 is positive or negative. The other two roots are of no interest to us here. It is relatively easy to solve (34) for v^* using standard iterative procedures, and from (33) it follows that

$$r^* = -\nu_3/v^* \ . \tag{35}$$

With r^* and v^* thus determined, the estimates m_1^*, m_2^*, θ_1^*, θ_2^* and α^* are given by (25) and (28) as in the general case, while σ^{*2} follows from the second equation of (31) as

$$\sigma^{*2} = v^* + \nu_2 \, . \tag{36}$$

A first approximation to r in the general case. In view of the relative ease with which estimates can be calculated when $\sigma_1 = \sigma_2$, we are thus provided with a simple procedure for obtaining initial approximations to r in the general case. With σ_1 assumed equal to σ_2, we obtain $v_{(o)}$ as the negative root of (34), and calculate $r_{(o)} = -\nu_3/v_{(o)}$, from (33). Unless the disparity between σ_1 and σ_2 is quite large, the resulting value, $r_{(o)}$, provides a satisfactory starting point from which we can iterate to the final estimate r^* in the general case.

Estimation in the Symmetric Cases

The compound normal distribution is symmetrical if (a) $\alpha = 1/2$ with $\sigma_1 = \sigma_2 = \sigma$, and if (b) $\theta_1 = \theta_2$. In the former instance the compound distribution has been shown to be bimodal when $\theta_2 - \theta_1 > 2\sigma$, but otherwise unimodal [8]. In the second case, the resultant distribution is always unimodal. In the limiting (trivial) case in which $\theta_1 = \theta_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$, the resultant distribution degenerates into a single normal distribution (μ, σ).

Symmetric case with $\alpha = 1/2$ and $\sigma_1 = \sigma_2$, $(\theta_1 \neq \theta_2)$. In this case, we need only estimate the three parameters σ, θ_1 and θ_2. From the second equation of (6), with $\alpha = 1/2$, it follows that $m_1 = -m_2$ and from the first equation of (31), $r = \beta = 0$. Consequently, from the second equation of (32), we have

$$2v^2 + k_4 = 0 \, . \tag{37}$$

With the vanishing of the odd central population moments, equation (37) might have been obtained as a special case of (34) with ν_3 replaced by zero, which in this instance is the appropriate estimate of μ_3.

It follows from (37) that

$$v^* = -\sqrt{-k_4/2}\ . \tag{38}$$

With v^* given by (38) and with $r = 0$, the required estimates follow from (25), (28), and (36) as

$$\left.\begin{aligned} &m_2^* = \sqrt{-v^*}, \quad m_1^* = -m_2^*\ , \\ &\theta_1^* = m_1^* + \bar{x}\ , \quad \theta_2^* = m_2^* + \bar{x}\ , \\ &\sigma^{*2} = v^* + \nu_2\ . \end{aligned}\right\} \tag{39}$$

<u>Symmetric case with $\theta_1 = \theta_2 = \theta$</u>. Since estimating equations (9) involve division by m_1 and m_2, and since it follows that $m_1 = m_2 = 0$ when $\theta_1 = \theta_2$, neither equations (9) nor subsequent equations derived from them are applicable here. We estimate θ from (7) as

$$\theta^* = \bar{x}\ . \tag{40}$$

With the vanishing of the odd central moments, estimation of the three remaining parameters, α, σ_1 and σ_2 necessitates use of the second, fourth, and sixth central moments. Applicable estimating equations accordingly become

$$\left.\begin{aligned} &\alpha(\sigma_1^2 - \nu_2) + (1-\alpha)(\sigma_2^2 - \nu_2) = 0\ , \\ &\alpha(3\sigma_1^4 - \nu_4) + (1-\alpha)(3\sigma_2^4 - \nu_4) = 0\ , \\ &\alpha(15\sigma_1^6 - \nu_6) + (1-\alpha)(15\sigma_2^6 - \nu_6) = 0\ . \end{aligned}\right\} \tag{41}$$

The first two of these equations follow directly from (8), and the third follows from (4) with $k = 6$ when μ_6 is estimated as ν_6.

From the first equation of (41), we have

$$\alpha = \frac{\nu_2 - \sigma_2^2}{\sigma_1^2 - \sigma_2^2}, \quad \text{and} \quad (1 - \alpha) = \frac{\sigma_1^2 - \nu_2}{\sigma_1^2 - \sigma_2^2} . \tag{42}$$

On substituting these values for α and $(1 - \alpha)$ into the second and third equations of (41), after considerable algebraic manipulations we obtain

$$\left.\begin{aligned} &(\sigma_1^2 - \nu_2)(\sigma_2^2 - \nu_2) = -k_4/3 , \\ &(\sigma_1^2 - \nu_2)(\sigma_2^2 - \nu_2)(\sigma_1^2 + \sigma_2^2 - 2\nu_2) = -k_6/15 , \end{aligned}\right\} \tag{43}$$

where k_4 and k_6 are the fourth and sixth order sample cumulants respectively; i.e.,

$$\left.\begin{aligned} &k_4 = \nu_4 - 3\nu_2^2 , \\ &k_6 = \nu_6 - 15\nu_4\nu_2 - 10\nu_3^2 + 30\nu_2^3 . \end{aligned}\right\} \tag{44}$$

Let

$$t_1 = \sigma_1^2 - \nu_2 \quad \text{and} \quad t_2 = \sigma_2^2 - \nu_2 , \tag{45}$$

and equations (43) become

$$\left.\begin{aligned} &t_1 t_2 = -k_4/3 , \\ &t_1 t_2 (t_1 + t_2) = -k_6/15 . \end{aligned}\right\} \tag{46}$$

It follows from (46) that t_1^* and t_2^* are roots of the quadratic equation

$$Y^2 - (k_6/5k_4)Y - (k_4/3) = 0 . \tag{47}$$

Accordingly

$$\left.\begin{aligned} t_1^* &= \frac{1}{2}\left[(k_6/5k_4) - \sqrt{(k_6/5k_4)^2 + (4k_4/3)}\right] , \\ t_2^* &= \frac{1}{2}\left[(k_6/5k_4) + \sqrt{(k_6/5k_4)^2 + (4k_4/3)}\right] . \end{aligned}\right\} \tag{48}$$

With t_1^* and t_2^* thus determined, it follows from (45) and from (41) that

$$\sigma_1^{*2} = t_1^* + \nu_2, \quad \sigma_2^{*2} = t_2^* + \nu_2 , \quad \alpha^* = -t_2^*/(t_1^* - t_2^*) , \tag{49}$$

and of course $\theta^* = \overline{x}$ as given in (40).

Determining Which Case Is Applicable

In the absence of a priori information concerning whether or not one of the symmetric special cases is applicable in lieu of the general case, the following criteria provide a basis for resolving this issue.

(a) If $\mu_3 = 0$ and if $\kappa_4 < 0$, the compound distribution is symmetric with $\sigma_1 = \sigma_2$ and with $\alpha = 1/2$.

(b) If $\mu_3 = 0$ and if $\kappa > 0$, the compound distribution is symmetric with $\theta_1 = \theta_2$.

(c) If $\mu_3 = 0$ and $\kappa_4 = 0$, then $\theta_1 = \theta_2$, $\sigma_1 = \sigma_2$, and the "resultant" distribution is in fact a single normal distribution.

Of course the third central moment is zero in all symmetrical distributions, and the converse likewise is true. Therefore, the foregoing statements can be verified by examining applicable expressions for the fourth cumulant, which is defined as $\kappa_4 = \mu_4 - 3\mu_2^2$. Using expressions from equations (6) for μ_4 and μ_2, the fourth population cumulant in the most general case follows as

$$\kappa_4 = 3\alpha(1-\alpha)[(\sigma_1^2 - \sigma_2^2) + (m_1^2 - m_2^2)]^2 - 2[\alpha m_1^4 + (1-\alpha)m_2^4] \, . \tag{50}$$

When $\alpha = 1/2$ and $\sigma_1 = \sigma_2$, this implies that $-m_1 = m_2$. With these values substituted into (50), we have

$$\kappa_4 = -[m_1^4 + m_2^4] < 0 \, . \tag{51}$$

When $\theta_1 = \theta_2$, this implies that $m_1 = m_2 = 0$, and in this instance (50) becomes

$$\kappa_4 = 3\alpha(1-\alpha)(\sigma_1^2 - \sigma_2^2)^2 \geq 0 \, . \tag{52}$$

When $\theta_1 = \theta_2$ and $\sigma_1 = \sigma_2$, it follows from (52) that in this limiting case $\kappa_4 = 0$, since (52) is applicable in all cases where $\theta_1 = \theta_2$.

In practical applications our classification problem is reduced to that of utilizing the sample statistics ν_3 and $k_4 = \nu_4 - 3\nu_2^2$ in choosing the most acceptable hypothesis from among the following alternatives: $H_{0:\mu_3=0,\kappa_4=0}$, $H_{1:\mu_3=0,\kappa_4<0}$, $H_{2:\mu_3=0,\kappa_4>0}$, $H_{3:\mu_3\neq 0}$.

An Illustrative Example

To illustrate the practical application of computational procedures developed in this paper, we consider a mixed sample obtained by combining two separate component samples consisting of 334 and 672 observations respectively from normal populations. For these two component samples, $\bar{x}_1 = 47.716$, $s_1^2 = 33.5663$, $N_1 = 334$, $\bar{x}_2 = 57.607$, $s_2^2 = 9.1790$ and $N_2 = 672$. For the resultant mixed sample, $n = N_1 + N_2 = 1006$, $\bar{x} = 54.32306$, $\nu_2 = s^2 = 38.9753$, $\nu_3 = -233.876$, $\nu_4 = 5365.13$, $\nu_5 = -67,821$ and $k_4 = 807.91$. In an effort to compensate for errors caused by grouping, Sheppard's corrections [11 (Vol. I, p. 71)] have been applied both in computing s_1^2 and s_2^2 for the separate components and in computing moments of the resultant mixed sample.

We obtain a first approximation to v by substituting the values given above for k_4 and ν_3 into equation (52) and solving for the negative root. By thus employing equation (52), we implicitly assume $\sigma_1 = \sigma_2$ for this first approximation. In subsequent approximations, we abandon this restriction and accordingly replace equation (52) with (29). For the first approximation our estimating equation, after making proper substitutions and simplifying, becomes

$$v^3 + 403.955\, v + 27{,}349 = 0 \; .$$

With the aid of a desk calculator, straightforward substitution quickly yields as a solution the first approximation $v_{(1)} = -25.7$. A first approximation to r then follows from (53) as

$$r_{(1)} = -(-233.876)/(-25.7) = -9.10 \; .$$

Using these values, first approximations to the basic parameters follow from (54), (25) and (28) as $\sigma^2_{1(1)} = \sigma^2_{2(1)} = 14.61$, $m_{1(1)} = -11.36$, $m_{2(1)} = 2.26$, and $\alpha_{(1)} = 0.166$. On substituting these values into the last equation of (6), we subsequently calculate $\mu_{5(1)} = -62{,}396$ which is to be compared with the sample value, $\mu_5 = -67{,}821$.

As a second approximation to r, we let $r_{(2)} = -4.50$ since further examination of the sample data with due regard for the shape of the histogram indicates that a value in the vicinity of -4 or -5 should be a good choice. This time we employ equation (29) rather than (34) in determining our new approximation to v. With $r = r_{(2)} = -4.50$, and with k_4 and ν_3 as previously given, equation (29) becomes

$$v^3 - 6.75\, v^2 - 297.7\, v + 9{,}116 = 0 \; .$$

On solving for the negative root of this equation, we find as our second approximation $v_{(2)} = -23.13$. Using the above values for $r_{(2)}$ and $v_{(2)}$, equations (25), (26), (27), and (28) yield as second approximations to the remaining parameters of interest, $m_{1(2)} = -7.56$, $m_{2(2)} = 3.06$, $\alpha_{(2)} = 0.288$, $\beta_{(2)} = -6.3705$, $\sigma^2_{1(2)} = 29.983$, and $\sigma^2_{2(2)} = 10.118$. When these values are substituted into the last equation of (6), we have $\mu_{5(2)} = -68,186$. For our next approximation to r, we interpolate linearly as indicated below.

r	μ_5
-9.10	-624×10^2
-4.82	-678×10^2
-4.50	-682×10^2

With $r_{(3)} = -4.82$, we subsequently calculate the remaining third approximations just as the second approximations were calculated, except that this time we retain additional significant digits. We accordingly obtain $v_{(3)} = -23.521$, $\beta_{(3)} = -6.52777$, $m_{1(3)} = -7.826$, $m_{2(3)} = 3.006$, $\alpha_{(3)} = 0.2775$, $\sigma^2_{1(3)} = 28.8156$, $\sigma^2_{2(3)} = 10.3167$, and finally $\mu_{5(3)} = -67,863$.

Extrapolation using the values of μ_5 calculated above with $r_{(2)} = -4.50$ and $r_{(3)} = -4.82$ yields $r_{(4)} = -4.87$ as the next and final five-moment estimate of r. Corresponding to this value for r, final five-moment estimates of other parameters of interest become $\theta_1^* = 46.456$, $\sigma_1^* = 5.352$, $\theta_2^* = 57.320$, $\sigma_2^* = 3.218$ and $\alpha^* = 0.2759$.

To calculate conditional minimum chi-square estimates, we require values of χ^2 for several values of r in the vicinity of the minimum. Since estimates of the basic parameters are already available for $r = -4.82$ and $r = -4.87$, we calculate expected frequencies and in turn χ^2 for these values of r and subsequently make the same calculations for $r = -4, -3, -2.5$ and -2, utilizing equations (29), (25), (26), (27) and (28) as previously described. Results of the pertinent calculations are summarized in Table IX which follows.

When the values of χ^2 from Table IX are plotted against corresponding values of r, it is readily observed that the minimum value ($\chi^2 = 0.72$) occurs when $r^{**} = -2.65$. With this value for r in (29), corresponding estimates for the remaining parameters of interest are computed as before. Accordingly as final conditional minimum chi-square estimates, we find $\theta_1^{**} = 48.304$, $\sigma_1^{**} = 6.042$, $\theta_2^{**} = 57.692$, $\sigma_2^{**} = 2.951$ and $\alpha^{**} = 0.3589$.

TABLE IX. SUMMARY OF ESTIMATES FOR VARIOUS TRIAL VALUES OF r

Parameters Estimated	r					
	-4.87	-4.82	-4	-3	-2.5	-2
v	-23.575	-23.521	-22.481	-20.098	-20.001	-19.022
m_1	-7.867	-7.826	-7.146	-6.312	-5.894	-5.475
m_2	2.997	3.006	3.146	3.312	3.394	3.475
β	-6.5535	-6.5278	-6.1344	-5.7287	-5.5644	-5.4317
σ_1^2	28.6397	28.8156	31.7464	35.2933	37.0327	38.7382
σ_1	5.352	5.368	5.635	5.941	6.086	6.224
σ_2^2	10.3513	10.3167	9.7791	9.0327	8.5705	8.0246
σ_2	3.218	3.212	3.127	3.006	2.928	2.833
θ_1	46.456	46.497	47.177	48.011	48.429	48.848
θ_2	57.320	57.329	57.469	57.635	57.717	57.798
α	0.2759	0.2775	0.3057	0.3441	0.3654	0.3883
μ_5	-67,821	-67,863	-68,657	-69,499	-69,835	-70,256
χ^2	3.20	3.06	1.58	0.80	0.74	0.98

For comparison, the expected frequencies based both on the five-moment estimates and the conditional minimum chi-square estimates are shown in Table X along with the observed frequencies.

Agreement between observed frequencies and expected frequencies based on either set of estimates is satisfactory. However, in view of the large sampling errors inherent in the fifth sample moment, it should come as no surprise to find that in this example, χ^2 for the five-moment estimates is substantially larger than that based on the conditional minimum chi-square estimates. For comparison, both sets of estimates calculated from the mixed sample are shown in Table XI along with corresponding estimates based on the individual components.

TABLE X. OBSERVED AND EXPECTED FREQUENCIES FOR 1006 OBSERVATIONS FROM A COMPOUND NORMAL DISTRIBUTION

Class Boundaries	Observed Frequencies	Expected Frequencies: Based on Five-Moment Estimates	Expected Frequencies: Based on Cond. Min. χ^2 Estimates
27.5 - 31.5	1	.7	.9
31.5 - 35.5	5	4.9	5.2
35.5 - 39.5	20	21.2	20.1
39.5 - 43.5	52	53.7	50.8
43.5 - 47.5	86	80.5	84.6
47.5 - 51.5	98	94.1	103.3
51.5 - 55.5	200	217.9	201.6
55.5 - 59.5	363	349.5	353.8
59.5 - 63.5	164	163.3	167.8
63.5 - 67.5	15	19.5	17.5
67.5 - 71.5	2	.6	.5
χ^2		3.20	0.72
d.f.		3	3
p		0.362	0.868

TABLE XI. COMPARISON OF ESTIMATES

Parameters	Component Estimates	Moment Estimates	Min. χ^2 Estimates
θ_1	47.716	46.456	48.304
θ_2	57.607	57.320	57.692
σ_1	5.794	5.352	6.042
σ_2	3.030	3.218	2.951
α	0.3320	0.2759	0.3589

In concluding, it is deemed appropriate to emphasize that the methods presented in this paper are recommended only with large samples. Furthermore, it is desirable that moments be calculated from the raw ungrouped data when possible. If the ungrouped data are not available, then at least grouping intervals should be relatively narrow in order to minimize errors in the higher moments from this source. When moments can only be computed from grouped data, it will usually be advisable to apply Sheppard's corrections.

ESTIMATION IN MIXTURES OF POISSON AND MIXTURES OF EXPONENTIAL DISTRIBUTIONS

by A. Clifford Cohen, Jr.

Summary

In the analysis of experimental data, many of the distributions encountered are the result of combining two or more separate component distributions. Estimation in these compound or mixed distributions is therefore of particular interest to aerospace scientists. Estimators are derived for the parameters of a compound Poisson distribution with probability density function

$$f(x) = \alpha \frac{e^{-\mu}\mu^{x}}{x!} + (1-\alpha)\frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\ldots$$

and for a compound exponential distribution with probability density function

$$f(x) = \alpha(1/\mu)e^{-x/\mu} + (1-\alpha)\,(1/\lambda)e^{-x/\lambda}, \quad x \geqq 0$$

where α is the proportionality factor ($0 \leqq \alpha \leqq 1$) and where μ and λ are component parameters. In addition to the more general case in which all parameters must be estimated from sample data, several special cases are considered in which one or more of the parameters are known in advance of sampling.

The research reported in this paper was performed under NASA Contract NAS8-11175 with the Aerospace Environment Office, Aero-Astrodynamics Laboratory, Marshall Space Flight Center, Huntsville, Alabama.

Introduction

Many of the distributions encountered in the analysis of experimental data are the result of combining two or more separate component distributions. Accordingly, estimation in these compound or mixed distributions is of particular interest to aerospace scientists. A previous paper [1] dealt with estimation in mixtures of two Poisson distributions; these previous results are extended here to include several special cases wherein one or more of the parameters of the compound Poisson distribution are known, and in addition analogous estimators are derived for the parameters of the compound exponential distribution.

The author wishes to acknowledge the assistance of Mr. Frank Clark for his work in establishing the IBM 7094 computer program described in the section entitled "Computational Procedures" and in the Appendix.

Mixtures of Two Poisson Distributions

The Probability Density Function. The probability density function of a compound distribution composed of two Poisson components with parameters μ and λ, respectively, combined in proportions α and $1 - \alpha$ may be written as

$$f(x) = \alpha \frac{e^{-\mu} \mu^x}{x!} + (1 - \alpha) \frac{e^{-\lambda} \lambda^x}{x!} \cdot \quad \begin{cases} x = 0, 1, 2, \ldots \\ 0 \leqq \alpha \leqq 1 \end{cases} \tag{1}$$

For convenience and without any loss of generality, we assume $\mu > \lambda$.

Three-Moment Estimators. The following estimating equations result from equating the first three factorial moments of a sample of size n to the corresponding theoretical moments.

$$\left.\begin{aligned} \alpha &= \frac{(\bar{x} - \lambda)}{(\mu - \lambda)} \\ \bar{x}\theta - \Gamma &= \nu_{[2]} \\ \bar{x}(\theta^2 - \Gamma) - \Gamma\theta &= \nu_{[3]} \end{aligned}\right\} , \tag{2}$$

where

$$\theta = \mu + \lambda \quad \text{and} \quad \Gamma = \mu\lambda , \tag{3}$$

and where the sample factorial moment $\nu_{[k]}$ is given by

$$\nu_{[k]} = \sum_{x=0}^{R} x(x-1)\ \dots\ (x-k+1)\frac{n_x}{n} , \tag{4}$$

in which R is the largest observed (sample) value of the random variable x, n_x is the sample frequency of x, and

$$n = \sum_{x=0}^{R} n_x .$$

For simplicity of notation, $\overline{x}$ has been written in place of $\nu_{[1]}$ for the first sample factorial moment.

On solving the last two equations of (2) simultaneously for Γ and θ, it follows that

$$\left.\begin{aligned} \theta^* &= \frac{\nu_{[3]} - \overline{x}\,\nu_{[2]}}{\nu_{[2]} - \overline{x}^2} \\ \Gamma^* &= \overline{x}\theta^* - \nu_{[2]} \end{aligned}\right\} , \tag{5}$$

where the asterisk (*) distinguishes estimators from the parameters being estimated. The required estimators of μ and λ follow as

$$\left.\begin{aligned} \mu^* &= \frac{1}{2}\ [\theta^* + \sqrt{\theta^{*2} - 4\Gamma^*}\ ] \\ \lambda^* &= \frac{1}{2}\ [\theta^* - \sqrt{\theta^{*2} - 4\Gamma^*}\ ] \end{aligned}\right\} . \tag{6}$$

These estimators are the two roots r_1 and r_2 of the quadratic equation

$$Y^2 - \theta^* Y + \Gamma^* = 0 \ , \tag{7}$$

where $\mu^* = r_1$ and $\lambda^* = r_2$, $(r_1 > r_2)$. The proportionality parameter α is estimated from the first equation of (2) as $\alpha^* = (\overline{x} - \lambda^*)/(\mu^* - \lambda^*)$.

The estimators given in equation (6) were originally derived by Rider [13], but he employed ordinary rather than factorial moments with the result that his derivations were somewhat complicated and his expressions for θ^* and Γ^* were more involved than those given here.

Estimators Based on the First Two Sample Moments and the Sample Zero-Frequency. It is well known that the higher sample moments are subject to appreciable sampling error, and in an effort to reduce errors from this source, the estimating equation based on the first two sample moments and the sample zero-frequency, was derived [14] as

$$\frac{\overline{x} - \lambda}{G(\lambda) - \lambda} = \frac{n_0/n - e^{-\lambda}}{e^{-G(\lambda)} - e^{-\lambda}} \ , \tag{8}$$

in which

$$G(\lambda) = \frac{\nu_{[2]} - \overline{x}\lambda}{\overline{x} - \lambda} \tag{9}$$

where n_0 is the sample zero-frequency. Equation (8) can be solved for λ^{**} using standard iterative procedures and, with λ^{**} thus determined, estimators of μ and α follow as

$$\left.\begin{aligned} \mu^{**} &= \frac{\nu_{[2]} - \overline{x}\lambda^{**}}{\overline{x} - \lambda^{**}} \\ \alpha^{**} &= \frac{\overline{x} - \lambda^{**}}{\mu^{**} - \lambda^{**}} \end{aligned}\right\} \ . \tag{10}$$

The double asterisk (**) distinguishes these estimators from the three-moment estimators and in turn from the parameters being estimated. Unfortunately, no simple procedure for solving equation (8) has been devised. However, a computer program based on iterative procedures described by Whittaker and Robinson [15 (Chap. VI)] has been developed (see Appendix) to solve equation (8), using as a first approximation the three-moment estimate of λ given by equation (6).

Estimation With Some Parameters Specified.

a. α Known

In this case, we need only estimate μ and λ; for this purpose, the first two equations of (2) may be written as

$$\left.\begin{aligned} \alpha &= \frac{\bar{x} - \lambda}{\mu - \lambda} \\ \bar{x}(\mu + \lambda) - \mu\lambda &= \nu_{[2]} \end{aligned}\right\} , \tag{11}$$

where θ and Γ have been replaced by their defining relations as given in equation (3).

With α known, we obtain the following quadratic equation in λ from the two equations of (11):

$$\lambda^2 - 2\bar{x}\lambda + \frac{\bar{x}^2 - \alpha\nu_{[2]}}{1 - \alpha} = 0 . \tag{12}$$

On solving equation (12)

$$\lambda^* = \bar{x} - \sqrt{\frac{\alpha(\nu_{[2]} - \bar{x}^2)}{1 - \alpha}} , \tag{13}$$

and from the first equation of (11)

$$\mu^* = [\overline{x} - \lambda^* (1 - \alpha)]/\alpha . \tag{14}$$

b. <u>α and μ Known</u>

In this case, λ may be estimated from the first equation of (11) as

$$\lambda^* = \frac{\overline{x} - \alpha\mu}{1 - \alpha} . \tag{15}$$

c. <u>α and λ Known</u>

In this case, it follows from equation (11) that

$$\mu^* = \frac{\overline{x} - (1 - \alpha)\lambda}{\alpha} . \tag{16}$$

d. <u>μ Known</u>

In this case, we may employ equations (11) to estimate α and λ. Accordingly, from the second equation of (11)

$$\lambda^* = \frac{\nu_{[2]} - \overline{x}\mu}{\overline{x} - \mu} , \tag{17}$$

and from the first equation of (11)

$$\alpha^* = \frac{\overline{x} - \lambda^*}{\mu - \lambda^*} . \tag{18}$$

e. <u>λ Known</u>

In this case, the second equation of (11) gives

$$\mu^* = \frac{\nu_{[2]} - \bar{x}\lambda}{\bar{x} - \lambda} , \tag{19}$$

and from the first equation of (11)

$$\alpha^* = \frac{\bar{x} - \lambda}{\mu^* - \lambda} . \tag{20}$$

Mixtures of Two Exponential Distributions

The Probability Density Function. In many respects the exponential distribution may be thought of as a continuous analog to the discrete Poisson distribution. In any event, estimating equations in mixtures of two exponential distributions quite closely parallel the estimating equations considered in the preceding section for mixtures of two Poisson distributions. Consider a compound exponential distribution with probability density function

$$f(x) = \alpha(1/\mu)e^{-x/\mu} + (1-\alpha)(1/\lambda)e^{-x/\lambda} . \quad \left\{ \begin{array}{l} x \geqq 0 \\ \mu > \lambda > 0 \\ 0 \leqq \alpha \leqq 1 \end{array} \right. \tag{21}$$

The nonessential restriction that $\mu > \lambda$ is imposed as a matter of convenience and without any loss of generality.

The kth noncentral moment of x is

$$m_k' = \int_0^\infty x^k f(x)\, dx = k! \ [\alpha\mu^k + (1-\alpha)\lambda^k] . \tag{22}$$

Accordingly, the first three noncentral moments are

$$\left. \begin{aligned} m_1' &= \alpha\mu + (1-\alpha)\lambda \\ m_2' &= 2[\alpha\mu^2 + (1-\alpha)\lambda^2] \\ m_3' &= 6[\alpha\mu^3 + (1-\alpha)\lambda^3] \end{aligned} \right\} . \tag{23}$$

Three-Moment Estimators. When the first three noncentral sample moments, designated ν_1', ν_2' and ν_3', respectively, with $\nu_1' = \bar{x}$, are equated to the theoretical moments of (23), we obtain the estimating equations

$$\left.\begin{aligned} &\bar{x} - \lambda = \alpha(\mu - \lambda) \\ &\frac{\nu_2'}{2} - \lambda^2 = \alpha(\mu^2 - \lambda^2) \\ &\frac{\nu_3'}{6} - \lambda^3 = \alpha(\mu^3 - \lambda^3) \end{aligned}\right\} . \tag{24}$$

These equations differ from the corresponding equations for mixed Poisson distributions only in that $\nu_2'/2$ and $\nu_3'/6$ have replaced the factorial moments $\nu_{[2]}$ and $\nu_{[3]}$ of the mixed Poisson distribution.

On eliminating α between the first and second and between the first and third equations of (24), we simplify to obtain

$$\left.\begin{aligned} &\bar{x}\theta - \Gamma = \frac{\nu_2'}{2} \\ &\bar{x}(\theta^2 - \Gamma) - \Gamma\theta = \frac{\nu_3'}{6} \end{aligned}\right\}, \tag{25}$$

which are completely analogous to the last two equations of (2) in the case of mixed Poisson distributions. Here, as in the Poisson case, θ and Γ are defined by equation (3). Accordingly, on solving the two equations of (25) simultaneously, we have as estimators of θ and Γ

$$\left.\begin{aligned} &\theta^* = \frac{\dfrac{\nu_3'}{6} - \bar{x}\dfrac{\nu_2'}{2}}{\dfrac{\nu_2'}{2} - \bar{x}^2} \\ &\Gamma^* = \bar{x}\theta^* - \frac{\nu_2'}{2} \end{aligned}\right\} , \tag{26}$$

which are analogous to equation (5) for the mixed Poisson distribution.

Finally, with θ^* and Γ^* determined from (26), μ^* and λ^* follow from equation (6) as in the Poisson case, and α^* follows from the first equation of (24) as

$$\alpha^* = \frac{\overline{x} - \lambda^*}{\mu^* - \lambda^*} \; . \tag{27}$$

Estimation With Some Parameters Specified.

a. α Known

We need only replace $\nu_{[2]}$ with $\nu_2'/2$ and the quadratic equation of (12) becomes, for the present case,

$$\lambda^2 - 2\overline{x}\lambda + \frac{\overline{x}^2 - \alpha \frac{\nu_2'}{2}}{1 - \alpha} = 0 \, . \tag{28}$$

Accordingly,

$$\left.\begin{aligned} \lambda^* &= \overline{x} - \sqrt{\frac{\alpha\left(\frac{\nu_2'}{2} - \overline{x}^2\right)}{1 - \alpha}} \\ \mu^* &= \frac{\overline{x} - \lambda^*(1 - \alpha)}{\alpha} \end{aligned}\right\} \; . \tag{29}$$

b. α and μ Known

In this case, the estimator for λ follows from the first equation of (24) as

$$\lambda^* = \frac{\overline{x} - \alpha\mu}{1 - \alpha} \; , \tag{30}$$

which is identical with the corresponding estimator, equation (15), in the Poisson case.

c. <u>α and λ Known</u>

In this case, it follows from the first equation of (24) that

$$\mu^* = \frac{\bar{x} - (1 - \alpha)\lambda}{\alpha} \quad . \tag{31}$$

d. <u>λ Known</u>

In this case, we need only replace $\nu_{[2]}$ in equation (19) with $\nu_2'/2$ and, accordingly,

$$\left.\begin{aligned} \mu^* &= \frac{\frac{\nu_2'}{2} - \bar{x}\lambda}{\bar{x} - \lambda} \\ \alpha^* &= \frac{\bar{x} - \lambda^*}{\mu - \lambda^*} \end{aligned}\right\} \quad . \tag{33}$$

Computational Procedures

The solution of the transcendental estimating equation (8) from the section entitled "Mixtures of Two Poisson Distributions" provides an interesting illustration of iterative numerical computational techniques described by Whittaker and Robinson [15]. To facilitate solution of equation (8), the denominator of the left side is interchanged with the numerator of the right side, and the resulting equation becomes

$$\frac{\bar{x} - \lambda}{n_0/n - e^{-\lambda}} = \frac{G(\lambda) - \lambda}{e^{-G(\lambda)} - e^{-\lambda}} \quad , \tag{34}$$

where $G(\lambda)$ remains as given by equation (9).

Equation (34) might be condensed to the form $L(\lambda) = R(\lambda)$ where

$$L(\lambda) = \frac{\bar{x} - \lambda}{n_0/n - e^{-\lambda}} \quad \text{and} \quad R(\lambda) = \frac{G(\lambda) - \lambda}{e^{-G(\lambda)} - e^{-\lambda}} . \tag{35}$$

The two functions $L(\lambda)$ and $R(\lambda)$ are essentially as plotted in Figure 1 below.

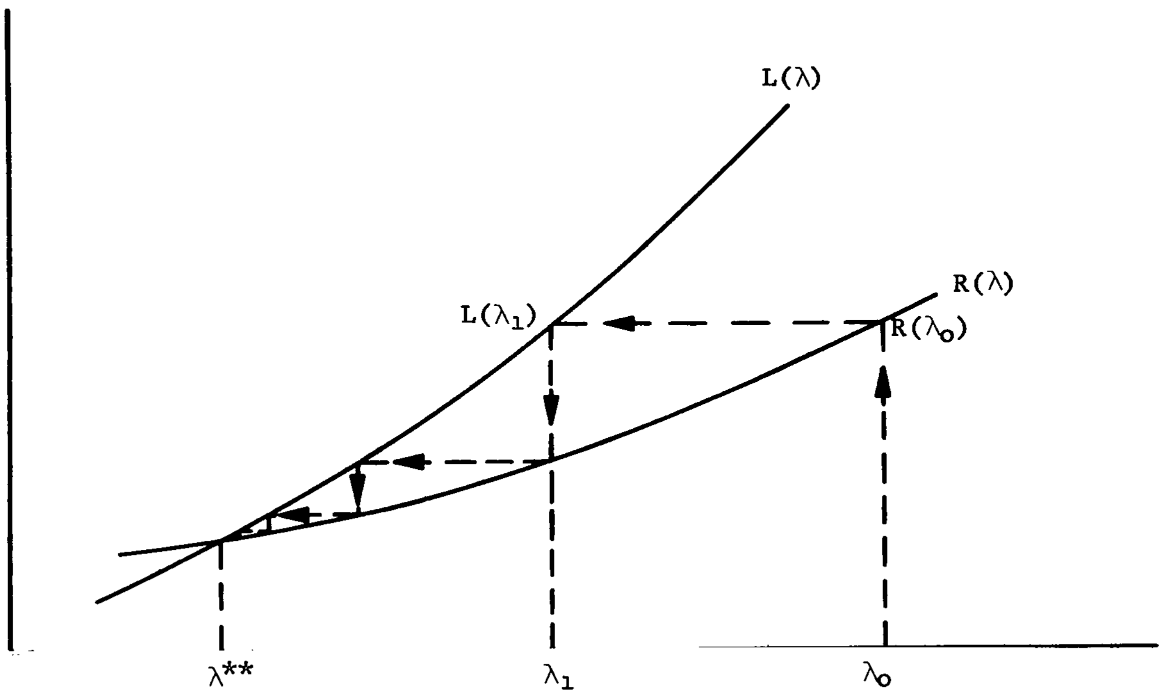


FIGURE 1. $L(\lambda)$ AND $R(\lambda)$

We begin with an initial approximation λ_0 and iterate toward the value λ^{**} as described by Whittaker and Robinson [15 (pp. 81-83)]. The three-moment estimate of λ given by equation (6) of the section entitled "Mixtures of Two Poisson Distributions" provides a satisfactory value for λ_0. This initial approximation is substituted into the second equation of (35) to obtain R_0, which is merely an abbreviated notation for $R(\lambda_0)$. We then solve the equation

$$L(\lambda_1) = R_0 \tag{36}$$

to obtain λ_1, the next approximation. This cycle is repeated as many times as necessary to attain the desired degree of accuracy. Equation (36) is itself a transcendental equation, though somewhat simpler in form than the original equation (34). It is amenable to solution by the Newton-Raphson method [15 (pp. 84-86)]. For the ith cycle of iteration, the equation corresponding to (36) becomes

$$L(\lambda_i) = \frac{\bar{x} - \lambda_i}{n_0/n - e^{-\lambda_i}} = R_{i-1} \,, \tag{37}$$

which may be written as

$$f(\lambda_i) = 0 \,,$$

where

$$f(\lambda_i) = \lambda_i - R_{i-1}\, e^{-\lambda_i} - C_{i-1} \tag{38}$$

and

$$C_{i-1} = (\bar{x} - R_{i-1}\, n_0/n) \,.$$

Equation (37) may be readily solved using the Newton-Raphson method, where $\lambda_{i:r+1}$, the $(r + 1)^{st}$ iterant to λ_i, is given by

$$\lambda_{i:r+1} = \lambda_{i:r} - \frac{f(\lambda_{i:r})}{f'(\lambda_{i:r})} \,.$$

The first derivative of $f(\lambda_i)$ follows from equation (38) as

$$f'(\lambda_i) = 1 + R_{i-1}\, e^{-\lambda_i} \,.$$

Accordingly,

$$\lambda_{i:r+1} = \lambda_{i:r} - \left[\frac{\lambda_{i:r} - R_{i-1} e^{-\lambda_{i:r}} - C_{i-1}}{1 + R_{i-1} e^{-\lambda_i}} \right] . \tag{39}$$

As an initial approximation $\lambda_{i:0}$ to λ_i, it will usually be satisfactory to let $\lambda_{i:0} = \lambda_{i-1}$. The Newton-Raphson iterative technique is continued through as many cycles as necessary to attain the desired accuracy in λ_i. More specifically, this procedure is terminated at the end of the r<u>th</u> cycle, the first cycle for which

$$|L_{i:r} - R_{i-1}| < \delta_1 ,$$

where δ_1 specifies the maximum permissible absolute value deviation. With λ_i thus determined, we calculate R_i, set up the new equation

$$L(\lambda_{i+1}) = R_i ,$$

and continue the primary routine through k cycles. The k<u>th</u> cycle is the first for which

$$|L_k - R_k| < \delta_2 , \tag{40}$$

where δ_2 specifies the maximum allowable absolute value deviation. The required estimate of λ is then

$$\lambda^{**} = \lambda_k .$$

Illustrative Examples

Mixed Poisson Distribution. To illustrate the application of his three-moment estimators, Rider [13] chose an example constructed by mixing equal proportions of two Poisson distributions with $\mu = 1.5$ and $\lambda = 0.5$, respectively. These data are as follows:

x	0	1	2	3	4	5	6	7
n_x	830	638	327	137	49	15	3	1

In summary, $n = 2000$, $n_0 = 830$, $\bar{x} = 0.9995$, $\nu_{[2]} = 1.248$ and $\nu_{[3]} = 1.734$. Direct substitution of these values into equations (5) and (6) yields the three-moment estimates

$$\mu^* = 1.4766563,$$

$$\lambda^* = 0.47765894,$$

$$\alpha^* = 0.52236479.$$

The above results differ slightly from those given by Rider caused apparently by small round-off errors in his calculations.

Estimates based on the first two sample moments and the sample zero-frequency, calculated by a computer program of the routine described in the preceding section, are

$$\mu^{**} = 1.4936,$$

$$\lambda^{**} = 0.4956, \text{ and}$$

$$\alpha^{**} = 0.5049.$$

These estimates are in much closer agreement with the actual population parameters $\mu = 1.5$, $\lambda = 0.5$, and $\alpha = 0.5$ than the three-moment estimates. Investigations are continuing with regard to the relative efficiency of the three-moment and the two-moment plus zero-frequency estimates; but at least in the present instance, where a large proportion of the population is in the zero class, the two-moment plus zero-frequency estimates seem to be preferred.

Mixed Exponential Distribution. To illustrate the application of estimators derived in this case, a sample of 2000 observations was selected from a mixed population constructed by combining two exponential distributions with $\mu = 2$, $\lambda = 1$, and $\alpha = 0.4$. Data for the sample selected are summarized as follows: $n = 2000$, $\bar{x} = 1.42$, $\nu_2' = 4.38$, and $\nu_3' = 21.6$.

Direct substitution of these data into equations (26), (6), and (27) yields as three-moment estimates:

$$\mu^* = 1.85,$$

$$\lambda^* = 1.02,$$

$$\alpha^* = 0.48.$$

Appendix

FIND - A Computer Program*

FIND is a Fortran IV computer program which calculates estimates for the parameters α, μ, and λ of a compound (mixed) Poisson distribution. These estimates are calculated from (a) the first three sample moments and (b) the first two sample moments and the sample zero-frequency.

In finding λ for the second case, the following equation is solved:

$$\frac{\bar{x} - \lambda}{G(\lambda) - \lambda} = \frac{n_0/n - e^{-\lambda}}{e^{-G(\lambda)} - e^{-\lambda}} ,$$

where

$$G(\lambda) = \frac{\nu_{[2]} - \bar{x}\lambda}{\bar{x} - \lambda} ,$$

and $\nu_{[2]}$, $\bar{x}$, and n_0/n are known constants. FIND makes use of the Newton-Raphson and geometrical iteration methods [15] in solving the equation.

FIND requires, for each data sample, input values for $\bar{x}$, n_0/n, $\nu_{[2]}$, and $\nu_{[3]}$, punched on a single card. Iteration continues through $k < 500$ cycles until the absolute error of equation (40) is less than 0.00001, i.e., until

$$|L_k - R_k| < 0.00001 .$$

If this criterion is not met when $k = 500$, the message "completed 500 iterations with no success" is given and the program stops. Should greater accuracy be required in the estimate of λ, appropriate change should be made in the source program card "TOL = 0.00 . . . ."

* The computer program presented was by Mr. Frank C. Clark.

FIND prints out the following:

1. Values of the index, i.

2. Values of λ in the Newton-Raphson iteration.

3. Values of

   ERROR = TEST 1 - TOL,

   where

   $$\text{TEST 1} = |L_{i:r} - R_{i-1}|.$$

4. α, μ, and λ based on the first three sample moments. (This value of λ is used as the first approximation in the Newton-Raphson process.)

5. α, μ, and λ based on the first two sample moments and the sample zero-frequency.

FIND (FORTRAN IV)

```
C.....ESTIMATION IN MIXTURES OF TWO POISSON DISTRIBUTIONS
      DIMENSION   LAM(4000)
      REAL MU,NUE2,NUE3,NON,LAM,L,LAMBDA,MU1
    1 READ(5,2)XBAR,NON,NUE2,NUE3
    2 FORMAT(4F10.5)
      THET = (NUE3-XBAR*NUE2)/(NUE2-(XBAR**2))
      CLAM = XBAR*THET-NUE2
      MU = (THET+SQRT(THET**2-4.0*CLAM))/2.0
      I=1
      LAM(I)= (THET-SQRT(THET**2-4.0*CLAM))/2.0
      N=0
      ALPHA1= (XBAR-LAM(I))/(MU-LAM(I))
      K=0
      G    = (NUE2-XBAR*LAM(I))/(XBAR-LAM(I))
      N=N+1
      R   = (G-LAM(I))/(EXP(-G)-EXP(-LAM(I)))
    9 C = (XBAR-NON*R)
   10 K=K+1
      LAM(I+1) = LAM(I)-((LAM(I)-R*EXP(-LAM(I))-C)/(1.0+R*EXP(-LAM(I))))
      L = (XBAR-LAM(I+1))/(NON-EXP(-LAM(I+1)))
      TOL = .00001
      TEST1 = ABS(L-R)
C     ......
   60 FORMAT(1H ,I5,5X,E15.8,E15.8)
      ERROR = TEST1 - TOL
      WRITE(6,60)I,LAM(I),ERROR
      IF (TEST1-TOL)20,15,15
   23 I=I+1
      GO TO 10
   20 G = (NUE2-XBAR*LAM(I+1))/(XBAR-LAM(I+1))
      R = (G-LAM(I+1))/(EXP(-G)-EXP(-LAM(I+1)))
      TEST2 = ABS(L-R)
      IF (TEST2-TOL)30,25,25
   24 I=I+1
      K=0
      GO TO 9
C     ......
   15 IF(500-K)22,22,23
   25 IF(500-N)22,22,24
   22 WRITE(6,28)
   28 FORMAT(42H1COMPLETED 500  ITERATIONS WITH NO SUCCESS)
      GO TO 100
   30 MU1      = (NUE2-XBAR*LAM(I+1))/(XBAR-LAM(I+1))
      LAMBDA   =  LAM(I+1)
      ALPHA2   =  (XBAR-LAM(I+1))/(MU1    -LAM(I+1))
C     ......
      WRITE(6,50)
   50 FORMAT(39H1ESTIMATES BASED ON FIRST THREE MOMENTS)
      WRITE(6,51)MU,LAM(1),ALPHA1
   51 FORMAT(10H0     MU = E15.8,10H LAMBDA = E15.8,9H ALPHA = E15.8)
      WRITE(6,52)
   52 FORMAT(74H0ESTIMATES BASED ON FIRST TWO SAMPLE MOMENTS AND THE ZER
     1O SAMPLE FREQUENCY)
      WRITE(6,53)MU1,LAMBDA,ALPHA2
   53 FORMAT(11H0      MU = E15.8,10H LAMBDA = E15.8,10H ALPHA =  E15.8)
      GO TO 1
  100 STOP
      END
```

ESTIMATION OF PARAMETERS IN COMPOUND WEIBULL DISTRIBUTIONS

by Lee W. Falls

Summary

The two-parameter Weibull distribution has been recognized as a useful model for survival populations associated with reliability studies and life testing experiments. In the analysis of atmospheric data, the distributions encountered are often a result of combining two or more component distributions. These compound distributions are consequently of interest to aerospace scientists. This paper presents a method for estimation of the parameters of a compound Weibull distribution with density function.

$$f(x) = \alpha f_1(x) + (1 - \alpha)\, f_2(x)\ , \qquad 0 < \alpha < 1$$

where

$$f_1(x) = \gamma_1\, \theta_1^{-1}\, x^{\gamma_1 - 1} \exp\left[-x^{\gamma_1}/\theta_1\right]\ , \quad x \geqq 0,\ \ \theta_1 > 0,\ \ \gamma_1 > 0$$

and

$$f_2(x) = \gamma_2\, \theta_2^{-1}\, x^{\gamma_2 - 1} \exp\left[-x^{\gamma_2}/\theta_2\right]\ , \quad x \geqq 0,\ \ \theta_2 > 0,\ \ \gamma_2 > 0.$$

The parameters required are α, the proportionality factor, γ_1, γ_2, θ_1 and θ_2. The most general case of estimation will be considered in addition to a number of special cases that may be of practical value.

Introduction

The Weibull distribution, derived in 1939 by W. Weibull, has been recognized as an appropriate model in reliability studies and life testing. Numerous methods for obtaining efficient estimates of the two parameters of this distribution have been outlined in recent years [16-18, 20].

In actual physical applications, however, a mixture of two Weibull distributions often seems to be a more desirable model. Distributions resulting from mixing two or more component distributions are designated as "mixed" or "compound." This situation is quite common in the analysis of atmospheric data and consequently is of interest to aerospace scientists.

Compound normal, Poisson and exponential distributions have been studied by A. C. Cohen, Jr. [2, 3, 19]. A method for estimating parameters of mixed distributions using sample moments has been outlined by Paul R. Rider [13] who considered compound Poisson, binomial, and a special case of the compound Weibull distribution. A graphical procedure for estimation of mixed Weibull parameters in life-testing of electron tubes is given by John H. K. Kao [20]. Although graphical methods have value for locating outliers, deriving initial estimates, and for determining whether the distribution is as hypothesized, for estimation purposes the analytic approach is probably superior.

This paper represents an attempt at estimating, by the method of sample moments, the five parameters of the compound Weibull distribution with density function

$$f(x) = \alpha f_1(x) + (1 - \alpha)\, f_2(x)\ , \quad 0 < \alpha < 1 \tag{1}$$

where

$$\left.\begin{aligned} f_1(x) &= \gamma_1\theta_1^{-1}x^{\gamma_1-1}\exp\,[-x^{\gamma_1}/\theta_1]\ , \quad x \geqq 0,\ \theta_1 > 0,\ \gamma_1 > 0 \\ f_2(x) &= \gamma_2\theta_2^{-1}x^{\gamma_2-1}\exp\,[-x^{\gamma_2}/\theta_2]\ , \quad x \geqq 0,\ \theta_2 > 0,\ \gamma_2 > 0 \end{aligned}\right\} . \tag{2}$$

The parameters involved are two scale parameters θ_1 and θ_2, two shape parameters γ_1 and γ_2, and the proportionality parameter α which expresses the probability that a given observation x_i comes from the population f_1.

The compound cumulative distribution function is defined

$$\left.\begin{aligned} F(x) = \alpha F_1(x) + (1 - \alpha)\, F_2(x) &= 1 - \alpha\exp\,[-x^{\gamma_1}/\theta_1] \\ &\quad - (1 - \alpha)\exp\,[-x^{\gamma_2}/\theta_2]\ . \end{aligned}\right\} \tag{3}$$

Figures 2 and 3 illustrate a generalized mixed Weibull probability density function and its corresponding distribution function.

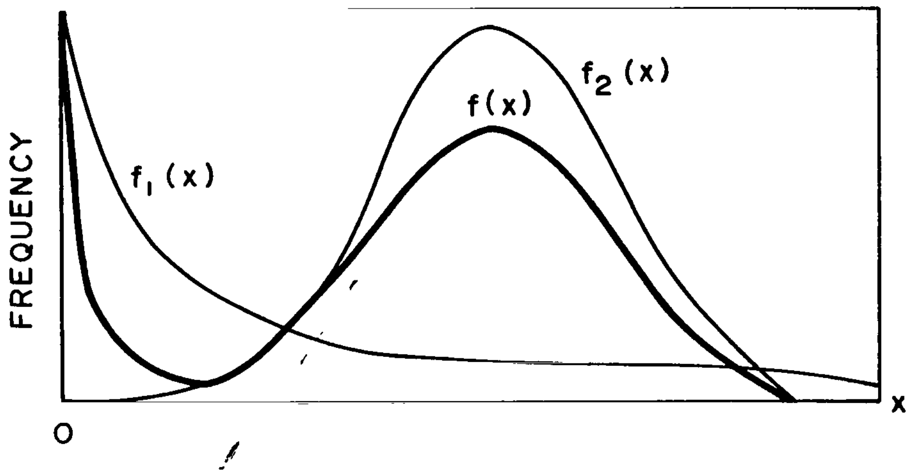


FIGURE 2. MIXED WEIBULL PROBABILITY DENSITY FUNCTION

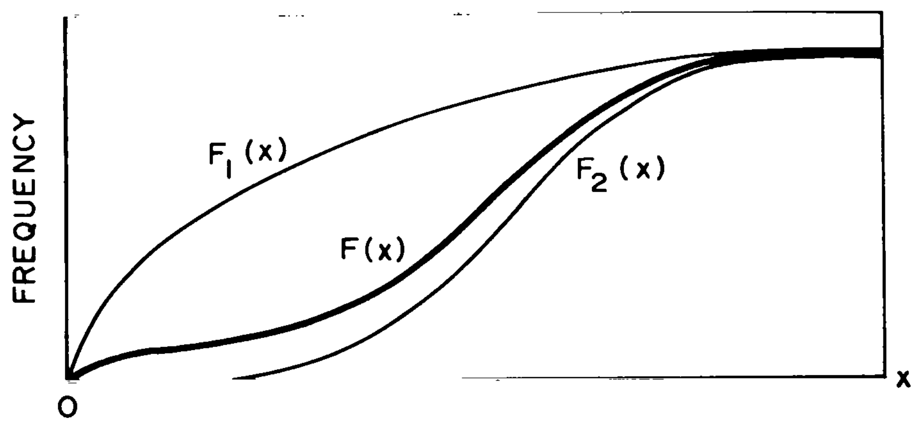


FIGURE 3. MIXED WEIBULL CUMULATIVE DISTRIBUTION FUNCTION

The most general case of estimation will be considered in which all five parameters must be estimated from the data. Also, a number of special cases will be investigated in which certain parameters are known in advance of sampling or are restricted in some manner. Included will be the special case where $\gamma_1 = 1$, i.e.,

$$f_1(x) = \theta_1^{-1} \exp [-x/\theta_1] , \tag{4}$$

which is the well-known exponential distribution. Estimating procedures are greatly simplified in these special cases as there are fewer sample moments involved in the estimating equations. Also, sampling errors are reduced because of the elimination of the need for higher order moments.

Estimation in the General Case

The r<u>th</u> theoretical moment about the origin of $f(x)$ is given by

$$\mu_r' = \alpha \int_0^\infty x^r f_1(x)\, dx + (1-\alpha) \int_0^\infty x^r f_2(x)\, dx , \tag{5}$$

where $f_1(x)$ and $f_2(x)$ are defined as in equations (2). The first five theoretical moments about the origin of (1) follow as

$$\left.\begin{aligned}
\mu_1' &= \alpha\, \theta_1^{1/\gamma_1} \Gamma\left(\frac{1}{\gamma_1} + 1\right) + (1-\alpha)\, \theta_2^{1/\gamma_2} \Gamma\left(\frac{1}{\gamma_2} + 1\right) \\
\mu_2' &= \alpha\, \theta_1^{2/\gamma_1} \Gamma\left(\frac{2}{\gamma_1} + 1\right) + (1-\alpha)\, \theta_2^{2/\gamma_2} \Gamma\left(\frac{2}{\gamma_2} + 1\right) \\
\mu_3' &= \alpha\, \theta_1^{3/\gamma_1} \Gamma\left(\frac{3}{\gamma_1} + 1\right) + (1-\alpha)\, \theta_2^{3/\gamma_2} \Gamma\left(\frac{3}{\gamma_2} + 1\right)
\end{aligned}\right\} \tag{6}$$

(Equation (6) concluded on following page.)

$$\left.\begin{aligned}
\mu_4' &= \alpha\ \theta_1^{4/\gamma_1}\ \Gamma\left(\frac{4}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{4/\gamma_2}\ \Gamma\left(\frac{4}{\gamma_2}+1\right) \\
\mu_5' &= \alpha\ \theta_1^{5/\gamma_1}\ \Gamma\left(\frac{5}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{5/\gamma_2}\ \Gamma\left(\frac{5}{\gamma_2}+1\right)
\end{aligned}\right\} \quad \text{(6) (Concluded)}$$

where Γ is the gamma function, i.e.,

$$\Gamma(k) = \int_0^\infty y^{k-1}\, e^{-y}\, dy\ .$$

Employing the technique of equating population moments to corresponding sample moments, the set of equations (6) becomes

$$\left.\begin{aligned}
m_1' &= \alpha\ \theta_1^{1/\gamma_1}\ \Gamma\left(\frac{1}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{1/\gamma_2}\ \Gamma\left(\frac{1}{\gamma_2}+1\right) \\
m_2' &= \alpha\ \theta_1^{2/\gamma_1}\ \Gamma\left(\frac{2}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{2/\gamma_2}\ \Gamma\left(\frac{2}{\gamma_2}+1\right) \\
m_3' &= \alpha\ \theta_1^{3/\gamma_1}\ \Gamma\left(\frac{3}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{3/\gamma_2}\ \Gamma\left(\frac{3}{\gamma_2}+1\right) \\
m_4' &= \alpha\ \theta_1^{4/\gamma_1}\ \Gamma\left(\frac{4}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{4/\gamma_2}\ \Gamma\left(\frac{4}{\gamma_2}+1\right) \\
m_5' &= \alpha\ \theta_1^{5/\gamma_1}\ \Gamma\left(\frac{5}{\gamma_1}+1\right) + (1-\alpha)\ \theta_2^{5/\gamma_2}\ \Gamma\left(\frac{5}{\gamma_2}+1\right)
\end{aligned}\right\} \quad (7)$$

where m_i' $(i = 1, 2, \ldots, 5)$ is the $i\underline{th}$ noncentral moment of the sample.

The set of equations (7) is a system of five equations which must be solved simultaneously for estimates of the five parameters α, θ_1, θ_2, γ_1 and γ_2. For convenience in handling these equations, we will make the following transformations where necessary in this paper. (This notation will be used unless stated otherwise.)

Let

$$\left.\begin{array}{lll} v = \theta_1^{1/\gamma_1} & \beta_1 = \Gamma\left(\frac{1}{\gamma_1}+1\right) & \psi_1 = \Gamma\left(\frac{1}{\gamma_2}+1\right) \\ n = \theta_2^{1/\gamma_2} & \beta_2 = \Gamma\left(\frac{2}{\gamma_1}+1\right) & \psi_2 = \Gamma\left(\frac{2}{\gamma_2}+1\right) \\ & \beta_3 = \Gamma\left(\frac{3}{\gamma_1}+1\right) & \psi_3 = \Gamma\left(\frac{3}{\gamma_2}+1\right) \end{array}\right\} \quad (8)$$

Thus, the first three equations of (7) become

$$m_1' = \alpha v \beta_1 + (1-\alpha)\, n\psi_1 \quad (9)$$

$$m_2' = \alpha v^2 \beta_2 + (1-\alpha)\, n^2\psi_2 \quad (10)$$

$$m_3' = \alpha v^3 \beta_3 + (1-\alpha)\, n^3\psi_3 \, . \quad (11)$$

Solving (9) for v, substituting this expression into (10), and then solving for n yields

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} \quad (12)$$

Substituting the expression for n from (12) back into (9) and solving for v gives

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1} \left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} \right]. \quad (13)$$

Upon substituting the expression for n from (12) and the expression for v from (13) into equation (11), we have one equation in the three unknowns α, γ_1 and γ_2. At this point, it is obvious that explicit expressions for the unknown parameters α, γ_1 and γ_2 cannot be obtained. Therefore, it is suggested that the following procedure be used to obtain a graphical estimate of α, the proportionality parameter. This method is essentially that of Kao [20] and is based upon the fact that a simple Weibull cumulative distribution becomes a straight line in ln versus ln-ln coordinates. This method of estimating α will produce a relatively small error in the estimating procedure since α is limited to the range $0 < \alpha < 1$.

a. Plot the sample cumulative distribution function for the mixed data on special ln versus ln-ln paper and visually fit a curve (called Weibull plot) among these points.*

b. Starting at each end of the Weibull plot, draw two tangent lines and denote them by $\widehat{\alpha F_1}$ and $\widehat{(1-\alpha) F_2}$ which are estimates of $\alpha F_1(x)$ and $(1-\alpha) F_2(x)$, respectively.

c. At the intersection of $\widehat{(1-\alpha) F_2}$ with the upper borderline drop a vertical line whose intersection with $\widehat{\alpha F_1}$ as read from the percent scale gives our estimate of α.

See Figure 5 under the section entitled "An Illustrative Example" for an illustration of this method.

Once an estimate of α has been determined graphically, solve equation (11) for γ_1 and γ_2 by the following iterative procedure. This procedure is taken from Cohen [3] and is a modified Newton-Raphson method.

---

*Special Weibull graph paper is available from Cornell University, Ithaca, New York.

Assume a value for γ_1 and solve equation (11) for a first approximation to γ_2. These first approximations can be substituted into (12) and (13) to obtain first approximations to θ_1 and θ_2. The first set of approximations is then introduced into the fourth equation of (6) to approximate the fourth noncentral theoretical moment, μ_4'.

Let $\gamma_{1(i)}$ denote the ith approximation to γ_1 and let $\mu'_{4(i)}$ denote the ith approximation (corresponding to $\gamma_{1(i)}$) to μ_4'. It should be relatively easy to find approximations $\gamma_{1(i)}$ and $\gamma_{1(i+1)}$ such that the sample moment m_4' is in the interval $[\mu'_{4(i)}, \mu'_{4(i+1)}]$. Once the interval between $\gamma_{1(i)}$ and $\gamma_{1(i+1)}$ has been narrowed sufficiently, the required estimate γ_1 can be obtained by a simple linear interpolation as indicated below.

γ_1	μ_4'
$\gamma_{1(i)}$	$\mu'_{4(i)}$
γ_1	m_4'
$\gamma_{1(i+1)}$	$\mu'_{4(i+1)}$

The required estimate of γ_2 can subsequently be obtained from equation (11). Once γ_1 and γ_2 have been determined by equation (11), estimates for θ_1 and θ_2 are obtained from equations (13) and (12), respectively.

Unfortunately, the quadratic solutions in equations (12) and (13) result in more than one set of estimates. The problem of non-unique sets of estimates was considered by Karl Pearson [4] and A. C. Cohen, Jr. [3] in connection with mixtures of two normal distributions. Pearson suggested choosing the set of estimates which gives closest agreement between the sixth sample moment and the sixth theoretical moment after equating the first five sample moments to the corresponding theoretical moments. This procedure is followed for all acceptable sets of estimates.

Cohen [3] suggests, as an alternate procedure for resolving the problem of multiple sets of estimates, that we might choose the set of estimates which produces the smallest Chi-Square Index of Dispersion when observed frequencies are compared with expected frequencies.

In the general case of estimation considered here, we are concerned with a mixture of two Weibull distributions where the proportionality factor α is estimated graphically from the cumulative frequencies and the four remaining parameters are estimated by equating the first four sample moments to corresponding theoretical moments. When confronted with more than one set of acceptable estimates, we adopt Pearson's suggested procedure and choose the set which produces the closest agreement between the fifth non-central moment of the sample m_5' and the theoretical moment μ_5' given by the final equation of (6).

The calculations described above may be carried out iteratively with relative ease using the computer program included in this paper as an Appendix. First approximations to initiate the iterative process may be obtained using a graphical method such as that of Kao [20].

Estimation in Special Cases

A number of special cases that may be of practical value in which certain parameters are known or are restricted in some manner are considered.

$\underline{\theta_1 \text{ Known}}$. With θ_1 known, we must estimate the parameters α, θ_2, γ_1 and γ_2 only. If we let $v = \theta_1^{1/\gamma_1}$, equations (9), (10) and (11) become

$$m_1' = \alpha\ \theta_1^{1/\gamma_1} \beta_1 + (1 - \alpha) n\psi_1 , \tag{14}$$

$$m_2' = \alpha\ \theta_1^{2/\gamma_1} \beta_2 + (1 - \alpha) n^2\psi_2 , \tag{15}$$

$$m_3' = \alpha\ \theta_1^{3/\gamma_1} \beta_3 + (1 - \alpha) n^3 \psi_3 . \tag{16}$$

Solving (14) for α gives

$$\alpha = \frac{m_1' - n\psi_1}{\theta_1^{1/\gamma_1} \beta_1 - n\psi_1} . \tag{17}$$

Inserting this expression for α into equation (15) and solving for n, we obtain after considerable algebraic manipulation

$$n = \frac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2} . \tag{18}$$

Substituting this expression for n back into (17) gives

$$\alpha = \frac{m_1' - \psi_1\left[\dfrac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2}\right]}{\theta_1^{1/\gamma_1}\beta_1 - \psi_1\left[\dfrac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2}\right]} \tag{19}$$

Now, upon substituting the expression for n from (18) and the expression for α from (19) into equation (16), we have an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative procedure described in the general case. Once γ_1 and γ_2 have been determined, we obtain our estimates for θ_2 and α from equations (18) and (19), respectively. As in the general case, the positive and negative roots resulting from the quadratic solution in (18) unfortunately results in more than one set of estimates. As before, it is suggested that the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve be used.

$\underline{\theta_2 \text{ Known}}$. With θ_2 known we need only estimate α, θ_1, γ_1 and γ_2. If we let $n = \theta_2^{1/\gamma_2}$, equations (9), (10) and (11) become

$$m_1' = \alpha v\beta_1 + (1 - \alpha)\ \theta_2^{1/\gamma_2}\ \psi_1 , \tag{20}$$

$$m_2' = \alpha v^2\beta_2 + (1 - \alpha)\ \theta_2^{2/\gamma_2}\ \psi_2 , \tag{21}$$

$$m_3' = \alpha v^3\beta_3 + (1 - \alpha)\ \theta_2^{3/\gamma_2}\ \psi_3 . \tag{22}$$

Solving (20) for α gives

$$\alpha = \frac{m_1' - \theta_2^{1/\gamma_2}\, \psi_1}{v\beta_1 - \theta_2^{1/\gamma_2}\, \psi_1} \ . \tag{23}$$

Inserting this expression for α into equation (21) and solving for v, we obtain after considerable algebraic manipulation

$$v = \frac{m_2'\beta_1 - n^2\beta_1\psi_2 \pm \sqrt{n^4\beta_1^2\psi_2^2 - 2n^2m_2'\beta_1^2\psi_2 + (m_2')^2\beta_1^2 - 4m_1'm_2'n\psi_1\beta_2 + 4m_2'n^2\psi_1^2\beta_2 + 4(m_1')^2n^2\beta_2\psi_2 - 4m_1'n^3\psi_1\psi_2\beta_2}}{2m_1'\beta_2 - 2n\psi_1\beta_2} \ . \tag{24}$$

Substituting this expression for v back into (23) gives

$$\alpha = \frac{m_1' - \theta_2^{1/\gamma_2}\psi_1}{\beta_1\left[\dfrac{m_2'\beta_1 - n^2\beta_1\psi_2 \pm \sqrt{n^4\beta_1^2\psi_2^2 - 2n^2m_2'\beta_1^2\psi_2 + (m_2')^2\beta_1^2 - 4m_1'm_2'n\psi_1\beta_2 + 4m_2'n^2\psi_1^2\beta_2 + 4(m_1')^2n^2\beta_2\psi_2 - 4m_1'n^3\psi_1\psi_2\beta_2}}{2m_1'\beta_2 - 2n\psi_1\beta_2}\right] - \theta_2^{1/\gamma_2}\psi_1} \ . \tag{25}$$

Now, upon substituting the expression for v from (24) and the expression for α from (25) into equation (22), we obtain an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative process described previously. Once γ_1 and γ_2 are determined, we obtain our estimates for θ_1 and α from equations (24) and (25). As in the case of θ_1 known, the positive and negative roots resulting from the quadratic solution in (24) gives more than one set of estimates. Again, it is suggested that we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

$\underline{\gamma_1 \text{ Known}}$. If γ_1 is known we must estimate α, θ_1, θ_2 and γ_2 only. Solving equation (9) for v gives

$$v = \frac{m_1' - (1 - \alpha)\ n\psi_1}{\alpha\beta_1} \ . \tag{26}$$

Inserting this expression for v into equation (10) and solving for n, we obtain

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} . \tag{27}$$

Substituting this expression for n back into (26) gives

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1}\left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}\right] . \tag{28}$$

Upon substituting the expression for n from (27) and the expression for v from (28) into equation (11), we obtain an equation in the two unknowns α and γ_2 which may be solved by the iterative procedure described in the general case. With α and γ_2 determined we may solve equations (27) and (28) for θ_2 and θ_1, respectively. As before, the positive and negative roots which result from the quadratic solution in (27) give more than one set of estimates. Again, we choose the set of estimates which gives the closest agreement to the fifth noncentral moment of the sample.

An alternate method for estimation in this case would be to estimate α graphically as in the general case and then solve equation (11) for γ_2 after the substitution of the expression for n from (27) and the expression for v from (28) into equation (11). As before, θ_2 and θ_1 would then be obtained from equations (27) and (28).

<u>γ_2 Known</u>. With γ_2 known, we must estimate α, θ_1, θ_2 and γ_1 only. As in the case, γ_1 known, solving equation (9) for v gives

$$v = \frac{m_1' - (1-\alpha)\, n\psi_1}{\alpha\beta_1} . \tag{29}$$

Inserting this expression for v into equation (10) and solving for n, we obtain

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} . \tag{30}$$

When we substitute this expression for n back into (29), we get

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1}\left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}\right] . \tag{31}$$

Upon inserting the expression for n from (30) and the expression for v from (31) into equation (11), we obtain an equation in the two unknowns α and γ_1 which may be solved by the iterative procedure described previously. With α and γ_1 determined in this manner, we now solve equations (30) and (31) for θ_2 and θ_1, respectively. As before, we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

As in the case of γ_1 known, an alternate method for estimation would be to estimate α, the proportionality parameter graphically, and then solve equation (11) directly for γ_1 after the substitution of the expression for n from (30) and the expression for v from (31) into equation (11).

$\underline{\gamma_1 = 1}$. This is a special case of γ_1 known. Thus, the case is reduced to mixing an exponential distribution with a Weibull distribution where $f_1(x)$ in equations (2) is an exponential distribution and $f_2(x)$ is a Weibull distribution. We need only estimate α, γ_2, θ_1 and θ_2.

With $\gamma_1 = 1$, equations (9), (10) and (11) become

$$m_1' = \alpha v + (1-\alpha)\, n\psi_1 \tag{32}$$

$$m_2' = 2\alpha v^2 + (1-\alpha)\, n^2\psi_2 \tag{33}$$

$$m_3' = 6\alpha v^3 + (1 - \alpha)\ n^3\psi_3 \ , \tag{34}$$

where the only change in notation from previous cases is $v = \theta_1$. Solving equation (32) for v gives

$$v = \frac{m_1' - (1 - \alpha)\ n\psi_1}{\alpha} \quad . \tag{35}$$

Substituting this expression for v into equation (33) and solving for n gives

$$n = \frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2(1-\alpha)^2\psi_1^2 + \alpha(1-\alpha)\psi_2} \quad . \tag{36}$$

Inserting this expression for n back into (35) gives

$$v = \frac{m_1'}{\alpha} - \psi_1 \left[\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2\alpha(1-\alpha)\psi_1^2 + \alpha^2\psi_2}\right] . \tag{37}$$

Now, upon substituting the expression for n from (36) and the expression for v from (37) into equation (34), we obtain an equation in the two unknowns α and γ_2 which we may solve using the iterative procedure described in previous cases. With minor simplifications, equation (34) becomes

$$m_3' = 6\alpha\left[\frac{m_1'}{\alpha} - \psi_1\left(\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2\alpha(1-\alpha)\psi_1^2 + \alpha^2\psi_2}\right)\right]^3 + \tag{38}$$

$$(1-\alpha)\psi_3\left[\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2(1-\alpha)^2\psi_1^2 + \alpha(1-\alpha)\psi_2}\right]^3 .$$

Once α and γ_2 have been determined from equation (38) we may obtain our estimates for θ_2 and θ_1 from equations (36) and (37), respectively.

As in the other cases, we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

An alternate method of estimation would be to estimate α graphically as in the general case and then solve equation (38) directly for γ_2. As above, estimates for θ_2 and θ_1 would then be obtained from equations (36) and (37).

$\underline{\gamma_1 = \gamma_2 = \text{Unknown}}$. Changing our notation slightly, we will let $\gamma_1 = \gamma_2 = \gamma$. Thus, we must estimate α, γ, θ_1 and θ_2. Now, let

$$v = \theta_1^{1/\gamma} \qquad \beta_1 = \Gamma\left(\frac{1}{\gamma} + 1\right) \qquad \beta_3 = \Gamma\left(\frac{3}{\gamma} + 1\right)$$

$$n = \theta_2^{1/\gamma} \qquad \beta_2 = \Gamma\left(\frac{2}{\gamma} + 1\right) .$$

With this notation, equations (9), (10) and (11) become

$$m_1' = \alpha v \beta_1 + (1 - \alpha)\, n\beta_1 \tag{39}$$

$$m_2' = \alpha v^2 \beta_2 + (1 - \alpha)\, n^2\beta_2 \tag{40}$$

$$m_3' = \alpha v^3 \beta_3 + (1 - \alpha)\, n^3\beta_3 . \tag{41}$$

As before, solving equation (39) for v gives

$$v = \frac{m_1' - (1 - \alpha)\, n\beta_1}{\alpha\beta_1} . \tag{42}$$

Inserting this expression for v into equation (40) and subsequently solving for n, we have

$$n = \frac{m_1'(1-\alpha)\beta_2 \pm \sqrt{(1-\alpha)\beta_2\,\alpha[m_2'\beta_1^2 - (m_1')^2\beta_2]}}{(1-\alpha)\beta_1\beta_2} . \tag{43}$$

Substituting this expression for n back into (42) gives

$$v = \frac{m_1'}{\alpha\beta_1} - \left[\frac{m_1'(1-\alpha)\beta_2 \pm \sqrt{\alpha(1-\alpha)\alpha[m_2'\beta_1^2 - (m_1')^2\beta_2]}}{\alpha\beta_1\beta_2}\right] . \tag{44}$$

Upon substituting the expression for n from (43) and the expression for v from (44) into equation (41), we obtain an equation in the two unknowns α and γ which may be solved by the iterative process described in the general case. We may now solve equations (43) and (44) for θ_2 and θ_1. As in the other cases, the quadratic solution in equation (43) results in more than one set of estimates. Again, we use the set of estimates which gives the closest agreement to the fifth noncentral moment of the sample.

As before, an alternate solution would be to estimate α graphically and then solve the resulting equation (41) directly for γ.

In the event that α is known in advance of sampling we may solve equation (41) directly for γ and subsequently obtain θ_2 and θ_1 from (43) and (44).

$\underline{\gamma_1 = \gamma_2 = \text{Known}}$. If we let $\gamma_1 = \gamma_2 = \gamma$, the first three equations of (7) become

$$m_1' = \alpha\theta_1^{1/\gamma}\,\Gamma\left(\frac{1}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{1/\gamma}\,\Gamma\left(\frac{1}{\gamma}+1\right) \tag{45}$$

$$m_2' = \alpha\theta_1^{2/\gamma}\,\Gamma\left(\frac{2}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{2/\gamma}\,\Gamma\left(\frac{2}{\gamma}+1\right) \tag{46}$$

$$m_3' = \alpha\theta_1^{3/\gamma}\,\Gamma\left(\frac{3}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{3/\gamma}\,\Gamma\left(\frac{3}{\gamma}+1\right) . \tag{47}$$

Thus, we must estimate α, θ_1 and θ_2 only. For simplification, we will 'et

$$c_j = \frac{m_j'}{\Gamma\left(\frac{j}{\gamma}+1\right)} .$$

Now, equations (45), (46) and (47) become

$$\alpha\theta_1^{1/\gamma} + (1-\alpha)\,\theta_2^{1/\gamma} = c_1 \tag{48}$$

$$\alpha\theta_1^{2/\gamma} + (1-\alpha)\,\theta_2^{2/\gamma} = c_2 \tag{49}$$

$$\alpha\theta_1^{3/\gamma} + (1-\alpha)\,\theta_2^{3/\gamma} = c_3 . \tag{50}$$

Solving equation (48) for α we have

$$\alpha = \frac{c_1 - \theta_2^{1/\gamma}}{\theta_1^{1/\gamma} - \theta_2^{1/\gamma}} . \tag{51}$$

Substituting the expression for α from (51) into (49) and (50), we have after considerable algebraic manipulation the equations

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma} = c_1\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) - c_2 \tag{52}$$

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) = c_1\left(\theta_1^{2/\gamma} + \theta_1^{1/\gamma}\,\theta_2^{1/\gamma} + \theta_2^{2/\gamma}\right) - c_3 . \tag{53}$$

Inserting the expression for $\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}$ from (52) into the left side of equation (53), we obtain after simplifying

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}\,c_1 - c_2\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) + c_3 = 0\,. \tag{54}$$

Solving (54) for $\theta_1^{1/\gamma}$ gives

$$\theta_1^{1/\gamma} = \frac{\theta_2^{1/\gamma}\,c_2 - c_3}{\theta_2^{1/\gamma}\,c_1 - c_2}\quad. \tag{55}$$

Substituting this expression for $\theta_1^{1/\gamma}$ back into equation (52), we have after simplification the quadratic equation in $\theta_i^{1/\gamma}$

$$(c_2 - c_1^2)\,[\theta_i^{1/\gamma}]^2 - (c_3 - c_1c_2)\,\theta_i^{1/\gamma} + (c_1c_3 - c_2^2) = 0\,, \tag{56}$$

whose solution for θ_i is

$$\theta_i = \left[2(c_1^2 - c_2)^{-1}\left[(-c_3 + c_1c_2) \pm (c_3^2 - 6c_1c_2c_3 - 3c_1^2c_2^2 + 4c_1^3c_3 + 4c_2^3)^{1/2}\right]\right]^{\gamma}\,. \tag{57}$$

Without loss of generality, we may impose the restriction that $\theta_1 < \theta_2$. Thus, we obtain θ_1 and θ_2 from equation (57) using the negative and positive roots, respectively. Once we have determined θ_1 and θ_2, we obtain our estimate for α from equation (51).

<u>$\theta_1 = \theta_2 =$ Known</u>. If we let $\theta_1 = \theta_2 = \theta$, the first three equations of (7) become

$$m_1' = \alpha\theta^{1/\gamma_1}\beta_1 + (1-\alpha)\,\theta^{1/\gamma_2}\,\psi_1 \tag{58}$$

$$m_2' = \alpha\theta^{2/\gamma_1}\beta_2 + (1-\alpha)\,\theta^{2/\gamma_2}\,\psi_2 \tag{59}$$

$$m_3' = \alpha\theta^{3/\gamma_1}\beta_3 + (1-\alpha)\,\theta^{3/\gamma_2}\,\psi_3\ . \tag{60}$$

Thus, we must estimate the parameters α, γ_1 and γ_2 only. As previously, solving (58) for α gives us

$$\alpha = \frac{m_1' - \theta^{1/\gamma_2}\,\psi_1}{\theta^{1/\gamma_1}\beta_1 - \theta^{1/\gamma_2}\,\psi_1}\ . \tag{61}$$

Substituting this expression for α into equation (59), we have after simplififying

$$m_1'\left(\theta^{2/\gamma_1}\beta_2 - \theta^{2/\gamma_2}\,\psi_2\right) + m_2'\left(\theta^{1/\gamma_2}\,\psi_1 - \theta^{1/\gamma_1}\,\beta_1\right)$$

$$-\,\theta^{2/\gamma_1}\beta_2\,\theta^{1/\gamma_2}\,\psi_1 + \theta^{2/\gamma_2}\,\psi_2\,\theta^{1/\gamma_1}\,\beta_1 = 0\ . \tag{62}$$

Equation (62) is an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative procedure described for the general case of estimation. Once γ_1 and γ_2 are determined, we obtain our estimate for α from equation (61).

$\underline{\theta_1 = \theta_2 = \text{Unknown}}$. For this special case, it is suggested that we solve for α graphically and subsequently follow the procedure outlined for the general case of estimation.

An Illustrative Example

To illustrate the estimation procedure outlined in this paper for the general case, we will consider a sample of 2000 observations selected from a mixed population constructed by combining two Weibull distributions with $\gamma_1 = 2.0000$, $\theta_1 = 10.0000$, $\gamma_2 = 0.8000$, $\theta_2 = 1.0000$ and $\alpha = 0.8000$. The sample is summarized in Table XII. For the sample selected, $m_1' = 2.4708$, $m_2' = 8.6270$, $m_3' = 36.3408$, $m_4' = 174.9190$ and $m_5' = 935.3733$.

TABLE XII. A SAMPLE OF 2000 OBSERVATIONS FROM A MIXED WEIBULL POPULATION

CLASSES	CLASS MARKS	f_1	f_2	f	CUMULATIVE FREQUENCY IN PERCENT
0 - 0.5	0.25	40	175	215	10.75
0.5 - 1.0	0.75	113	78	191	20.30
1.0 - 1.5	1.25	170	47	217	31.15
1.5 - 2.0	1.75	205	30	235	42.90
2.0 - 2.5	2.25	216	20	236	54.70
2.5 - 3.0	2.75	206	14	220	65.70
3.0 - 3.5	3.25	180	10	190	75.20
3.5 - 4.0	3.75	147	7	154	82.90
4.0 - 4.5	4.25	112	5	117	88.75
4.5 - 5.0	4.75	80	4	84	92.95
5.0 - 5.5	5.25	54	3	57	95.80
5.5 - 6.0	5.75	34	2	36	97.60
6.0 - 6.5	6.25	20	1	21	98.65
6.5 - 7.0	6.75	11	1	12	99.25
7.0 - 7.5	7.25	6	1	7	99.60
7.5 - 8.0	7.75	3	1	4	99.80
8.0 - 8.5	8.25	1	1	2	99.90
8.5 - 9.0	8.75	1		1	99.95
9.0 - 9.5	9.25	1		1	100.00
		1600	400	2000	

In Table XII, f_1 = class frequencies from $f_1(x)$, f_2 = class frequencies from $f_2(x)$, and f = class frequencies from the resulting mixed distribution.

Figure 4 is a graph of the compound density function and its component distributions. Notice at this point that $\gamma \leqq 1$ produces a J-shaped function while $\gamma > 1$ produces a bell-shaped curve.

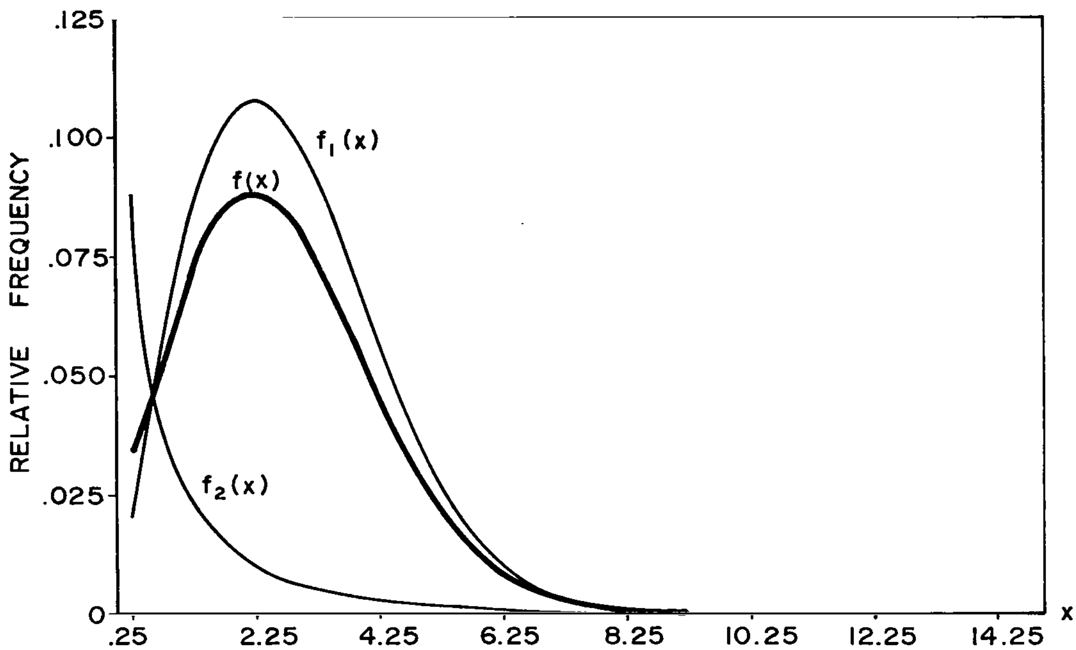


FIGURE 4. ILLUSTRATIVE EXAMPLE OF MIXED WEIBULL DENSITY FUNCTION

Employing the graphical technique described in the first section provides an estimate of α, the proportionality parameter, equal to 0. 80 as shown on Figure 5.

Once our estimate of α has been determined graphically, we solve equation (11) iteratively for first approximations to γ_1 and γ_2. Corresponding first approximations for θ_1 and θ_2 are obtained from equations (13) and (12), respectively. Each set of first approximations is introduced into the fourth equation of (6) to approximate the fourth noncentral theoretical moment, μ_4'. Each set of first approximations is also substituted into the final equation of (6) to approximate the fifth noncentral theoretical moment μ_5', as suggested by Pearson [4]. The set of estimates which gives the closest agreement between the fifth noncentral moment of the sample m_5' and the corresponding

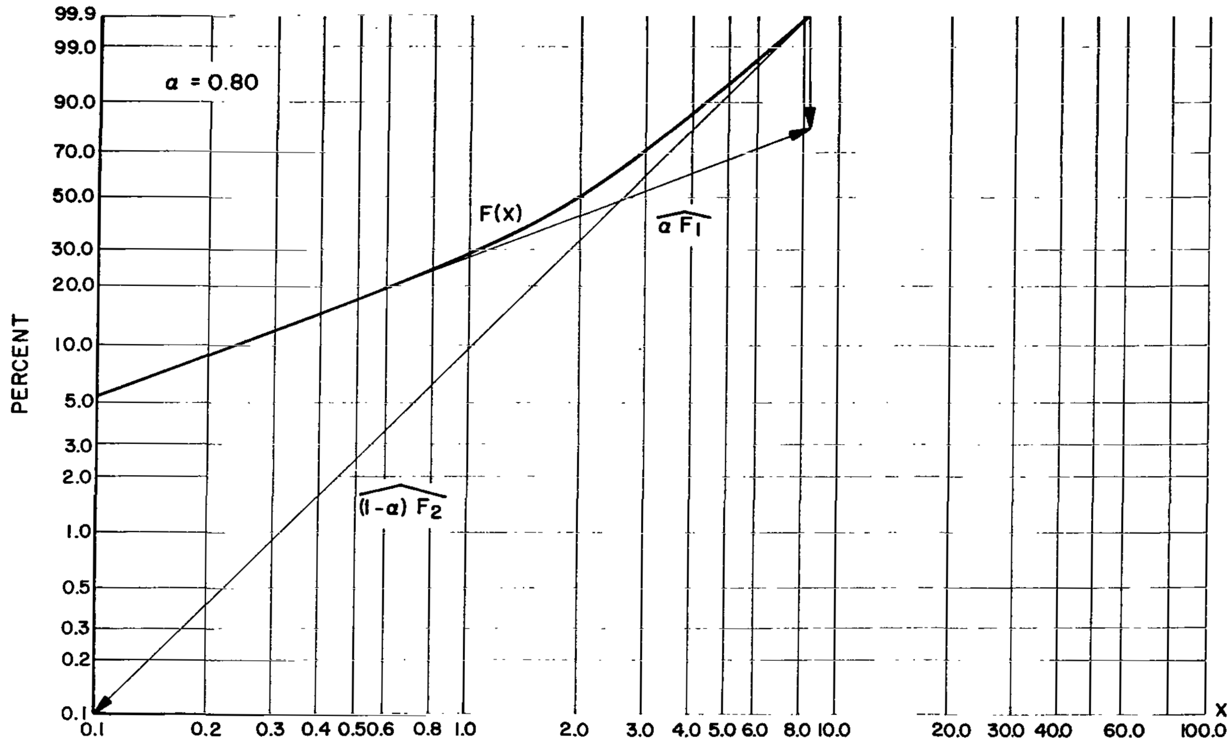


FIGURE 5. WEIBULL PLOT

"fitted" compound curve given by the final equation of (6) is the required set of estimates. Utilizing the computer program given in the Appendix, we find that our estimate of γ_1 lies between 1.90 and 2.10. The corresponding estimates for the remaining parameters, μ_4' and μ_5', are as follows:

γ_1	γ_2	θ_1	θ_2	μ_4'	μ_5'
1.90	2.30	11.52	1.29	343.8875	1926.6399
2.10	0.80	6.74	0.82	64.6006	287.5141

Our next approximation to γ_1 is obtained by simple linear interpolation as indicated below.

γ_1	μ_4'
1.90	343.8875
2.0210	174.9190
2.10	64.6006

Substituting $\gamma_1 = 2.0210$ into equation (11), solving for γ_2 and subsequently solving equations (13) and (12) for θ_1 and θ_2, we obtain $\gamma_2 = 0.8339$, $\theta_1 = 10.1496$ and $\theta_2 = 0.9929$. Introducing this set of approximations into the final equation of (6) gives $\mu_5' = 935.3646$. This value of μ_5' is in such close agreement with the corresponding sample moment $m_5' = 935.3733$ that we are justified in accepting this set of approximations as our final estimates. However, if further preciseness is desired, this iterative process may be continued to any desired degree of accuracy.

The computer program outlined in the Appendix gives all possible solutions to the estimating equations. Specifically, equations (12) and (13) produce four solutions resulting from the four combinations of the positive and negative signs prefixing the radicals; i.e., the combinations are (-,-), (-,+), (+,-) and (+,+). It was discovered that the computer program printout gave closest agreement between the theoretical moment μ_5' and the sample moment m_5' when the combination (-,-) was used.

Comparisons of expected with observed frequencies, along with a comparison of observed and expected distribution functions, are presented in Table XIII. It also seems appropriate to compare the observed frequencies for the mixed sample with these same frequencies assuming that the sample fits a simple Weibull distribution. This will prove or disprove that our mixed sample could be treated as a simple Weibull distribution. Cohen's maximum likelihood estimation procedure [16], was used to derive estimates for the parameters, and the resulting expected frequencies were obtained. Notations used in Table XIII are as follows:

f_o = observed frequencies for mixed data.

F_o = observed distribution function for mixed data.

f_e = expected frequencies using estimates derived in this report.

F_e = expected distribution function using estimates derived in this report.

f_{es} = expected frequencies assuming data fits a simple Weibull distribution.

F_{es} = expected distribution function assuming data fits a simple Weibull distribution.

TABLE XIII. OBSERVED AND EXPECTED FREQUENCIES FOR 2000 OBSERVATIONS FROM A MIXED WEIBULL DISTRIBUTION

CLASSES	f_o	f_e	f_{es}	F_o	F_e	F_{es}
0 - 0.5	215	211.0	143.3	0.1075	0.1055	0.0717
0.5 - 1.0	191	192.9	243.2	0.2030	0.2020	0.1932
1.0 - 1.5	217	219.1	271.9	0.3115	0.3116	0.3292
1.5 - 2.0	235	237.8	265.8	0.4290	0.4305	0.4621
2.0 - 2.5	236	238.9	239.9	0.5470	0.5499	0.5821
2.5 - 3.0	220	222.1	204.6	0.6570	0.6610	0.6843
3.0 - 3.5	190	191.7	166.8	0.7520	0.7569	0.7677
3.5 - 4.0	154	154.3	130.9	0.8290	0.8340	0.8332
4.0 - 4.5	117	116.2	99.4	0.8875	0.8921	0.8828
4.5 - 5.0	84	82.1	73.3	0.9295	0.9332	0.9195
5.0 - 5.5	57	54.6	52.6	0.9580	0.9605	0.9458
5.5 - 6.0	36	34.3	36.8	0.9760	0.9776	0.9642
6.0 - 6.5	21	20.4	25.2	0.9865	0.9878	0.9768
6.5 - 7.0	12	11.5	16.9	0.9925	0.9935	0.9852
7.0 - 7.5	7	6.2	11.1	0.9960	0.9966	0.9908
7.5 - 8.0	4	3.2	7.1	0.9980	0.9982	0.9943
8.0 - 8.5	2	1.6	4.5	0.9990	0.9990	0.9966
8.5 - 9.0	1	.8	2.8	0.9995	0.9994	0.9980
9.0 - 9.5	1	.4	1.7	1.0000	0.9997	0.9988

The agreement between observed frequencies for the sample and expected frequencies using the derived estimates is very good as shown in Table XIII. The corresponding observed and expected distribution functions are in very close agreement with the maximum absolute difference of 0.0050 occurring at the class 3.5 - 4.0. Comparing this value with the Kolmogorov-Smirnov statistic, we see that

$$D_{\alpha}^{N} = D_{0.01}^{2000} = 0.0364$$

gives an excellent "goodness-of-fit" at the 99-percent level of confidence.

Comparing F_o with F_{es} shows a maximum absolute difference in the distribution functions of 0.0358 occurring at the class 0 - 0.5. Since the Kolmogorov-Smirnov statistic

$$D_{\alpha}^{N} = D_{0.05}^{2000} = 0.0304 ,$$

this value of 0.0358 is sufficient to reject the hypothesis that our mixed distribution could be considered a simple Weibull distribution.

As an alternate goodness-of-fit test for agreement between observed frequencies and expected frequencies using the derived estimates, the χ^2 index was calculated and the results are as follows:

$$\chi^2 = 1.2669 \qquad \text{d.f.} = 10 \qquad P(\chi^2 > 1.2669 = 0.995) .$$

Thus, in consideration of the low χ^2 index of dispersion, we may conclude that we have an excellent fit for the chosen sample.

Conclusions

It is an accepted fact that the method of moments is not (except for distributions such as the normal, binomial, and Poisson) the most efficient procedure for estimating the parameters of a frequency distribution. Methods having maximum efficiency, such as the method of maximum likelihood, are more desirable. However, in the case of the mixed Weibull distribution with its five parameters, the maximum likelihood estimating equations are almost intractable.

The central, noncentral, and factorial moments of this distribution were investigated, and it was discovered that the noncentral moments possessed optimum characteristics for the development of estimating equations. A comparison of noncentral sample moments with the theoretical moments for the sample selected showed an error of 0.09 percent for the first moment, 0.44 percent for the second moment, 1.64 percent for the third moment, 4.94 percent for the fourth moment, and 12.73 percent for the fifth moment. This progressive increasing percentage of nonagreement between sample moments and theoretical moments illustrates the large sampling errors involved in the use of higher order moments. Using only the first three sample moments with their relatively low percentage of error in the estimating equations produced very good agreement between final estimates and the population parameters γ_1, γ_2, θ_1 and θ_2. Sheppard's corrections for grouped data were applied to the sample moments in order to increase this agreement, but produced no significant change in the results; therefore, the corrections were not used in the estimating equations.

This paper presents an estimating procedure that produced very good results for the sample chosen. The use of electronic digital computers makes the somewhat involved method practical and applicable to experiments in which the mixed Weibull distribution is the appropriate statistical model.

In the author's opinion, the estimating procedures outlined in this report warrant further investigation for increase in efficiency, and improvement and simplification of form for the estimating equations involved.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, March 29, 1967.
160-44-04-00-62

Appendix

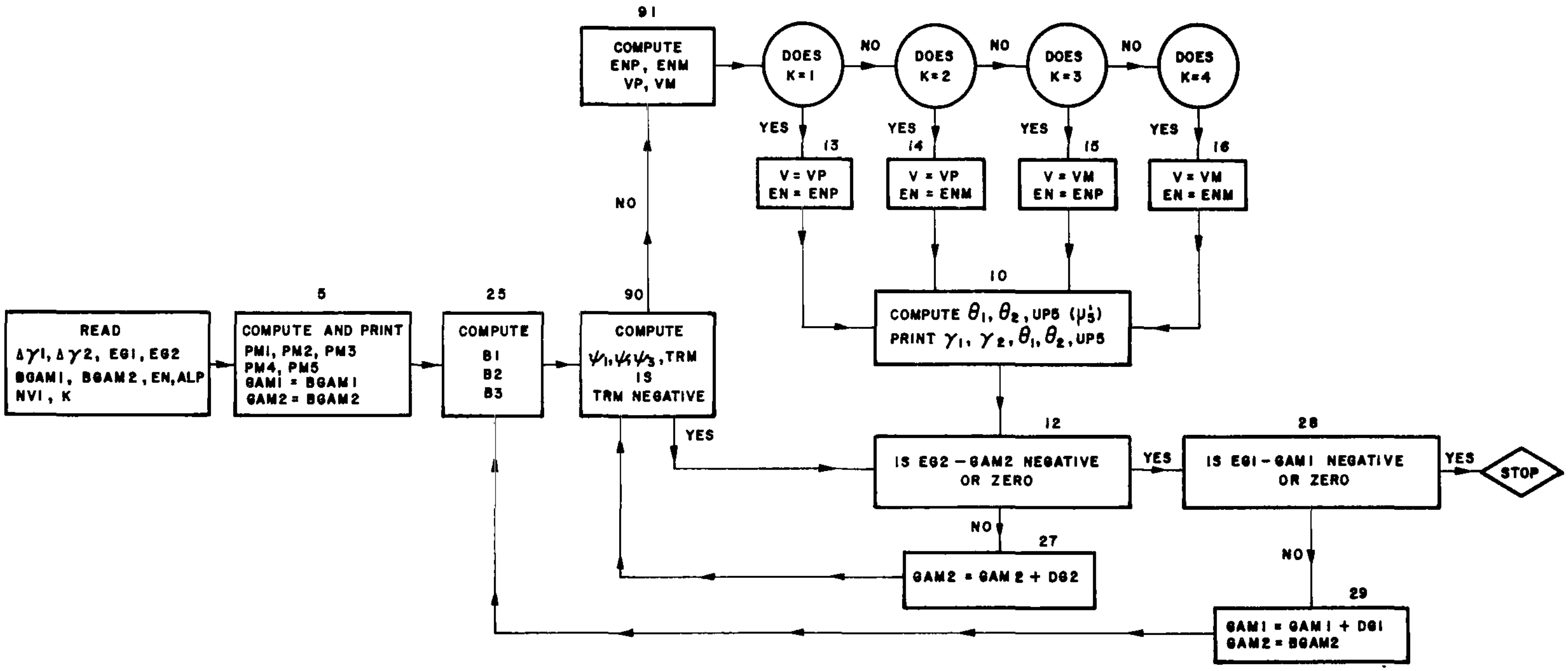
Computer Program for Estimating the Parameters of a Mixed Weibull Distribution

3200 FORTRAN

ESTIMATION IN MIXTURES OF TWO WEIBULL DISTRIBUTIONS

```
      DIMENSION XI(50),FI(50)
      READ(60,1)DG1,DG2,FG1,FG2
      READ(60,1)BGAM1,BGAM2,FN,ALP
    1 FORMAT(4F10.5)
      READ(60,2)NV1,K
    2 FORMAT(I3/I3)
      READ(60,3)(XI(I),I=1,NV1)
      READ(60,3)(FI(I),I=1,NV1)
    3 FORMAT(8F10.5)
      WRITE(61,4)
      PM1=0.0
      PM2=0.0
      PM3=0.0
      PM4=0.0
      PM5=0.0
      DO 5 I=1,NV1
      FR1=XI(I)
      FR2=FR1*XI(I)
      FR3=FR2*XI(I)
      FR4=FR3*XI(I)
      FR5=FR4*XI(I)
      PM1=PM1+FR1*FI(I)
      PM2=PM2+FR2*FI(I)
      PM3=PM3+FR3*FI(I)
      PM4=PM4+FR4*FI(I)
      PM5=PM5+FR5*FI(I)
    5 CONTINUE
      PM1=PM1/FN
      PM2=PM2/FN
      PM3=PM3/FN
      PM4=PM4/FN
      PM5=PM5/FN
      WRITE(61,70)PM1,PM2,PM3,PM4,PM5
      GAM1=BGAM1
      GAM2=BGAM2
   25 TRM=1.0/GAM1+1.0
      B1=GAMMA(TRM)
      TRM=2.0/GAM1+1.0
      B2=GAMMA(TRM)
      TRM=3.0/GAM1+1.0
      B3=GAMMA(TRM)
      WRITE(61,92)
   90 TRM=1.0/GAM2+1.0
      PSI1=GAMMA(TRM)
      TRM=2.0/GAM2+1.0
      PSI2=GAMMA(TRM)
      TRM=3.0/GAM2+1.0
      PSI3=GAMMA(TRM)
      FR=ALP*(1.0-ALP)*B1**2*PSI2
      FR1=(1.0-ALP)**2*PSI1**2*B2
      DEN=2.0*(FR+FR1)
      FR2=-4.0*PM1**2*FR*B2
      FR3=4.0*PM2*ALP*FR1*B1**2
      FR4=4.0*PM2*ALP**2*(1.0-ALP)*B1**4*PSI2
      TRM=FR2+FR3+FR4
      IF(TRM)12,91,91
   91 TRM1=SQRT(TRM)
      FR5=2.0*PM1*(1.0-ALP)*PSI1*B2
      FNP=(FR5+TRM1)/DEN
      FNM=(FR5-TRM1)/DEN
      ER6=(PM1-(1.0-ALP)*PSI1)/(ALP*B1)
      VP=ER6*FNP
      VM=ER6*FNM
      GO TO (13,14,15,16)K
   13 V=VP
      FN=FNP
      GO TO 10
   14 V=VP
      FN=FNM
      GO TO 10
   15 V=VM
      EN=ENP
      GO TO 10
   16 V=VM
      EN=FNM
      GO TO 10
   12 IF(FG2-GAM2-.000001)28,28,27
   27 GAM2=GAM2+DG2
      GO TO 90
   28 IF(FG1-GAM1-.00001)100,100,29
   29 GAM1=GAM1+DG1
      GAM2=BGAM2
      GO TO 25
   10 TH1=EXP(GAM1*ALOG(V))
      TH2=EXP(GAM2*ALOG(EN))
      ER=5.0/GAM1+1.0
      TRM=GAMMA(FR)
      FR=5.0/GAM2+1.0
      TRM1=GAMMA(FR)
      FR1=ALP*(TH1**(5.0/GAM1))
      FR2=(1.0-ALP)*(TH2**(5.0/GAM2))
      UP5=FR1*TRM+FR2*TRM1
      WRITE(61,36)GAM1,GAM2,TH1,TH2,UP5
      GO TO 12
  100 STOP
    4 FORMAT(1H1)
   70 FORMAT(3X,6HPM1   =F14.7,3X,6HPM2   =F14.7,3X,6HPM3   =F14.7,
     13X,6HPM4   =F14.7,3X,6HPM5   =F14.7/)
   92 FORMAT(//)
   36 FORMAT(3X,6HGAM1 =F15.8,3X,6HGAM2 =F15.8,3X,
     16HTH1   =F15.8,3X,6HTH2   =E15.8,3X,6HU5    =E15.8)
      END
```

```
      FUNCTION GAMMA(X)                                                  GAM00001
   75 FORMAT(53H GAMMA FUNCTION OF A NEGATIVE INTEGER IS NOT DEFINED.)   GAM00002
    5 IF(X) 10,24,15                                                     GAM00003
   10 N=-X                                                               GAM00004
      EN=-N-1                                                            GAM00005
      V=X-EN                                                             71M00006
      IF(V-1.)20, 80, 20                                                 GAM00007
   15 N=X                                                                GAM00008
      EN=N                                                               GAM00009
      V=X-EN                                                             GAM00010
   20 GAMMA=1.+V*(.42278434 +V*(.41184025  +V*(.081578219  +V*          GAM00011
     1(.074237908  +V*(-.00021090747  +V*(.010973696  +V*(-.002466748   GAM00012
     2+V*(.0015397681  -V*(.00034423420  -V*.00006771057   )))))))))    GAM00013
      IF(EN-2.) 37,25,30                                                 GAM00014
   24 GAMMA=1.                                                           GAM00015
   25 RETURN                                                             GAM00016
   30 N=N-1                                                              GAM00017
      DO 35 I=2,N                                                        GAM00018
      FI=I                                                               GAM00019
   35 GAMMA=GAMMA*(FI+V)                                                 GAM00020
      RETURN                                                             GAM00021
   37 N=2.-EN                                                            GAM00022
      DO 40 I=1,N                                                        GAM00023
      FI=2-I                                                             GAM00024
   40 GAMMA=GAMMA/(FI+V)                                                 GAM00025
      RETURN                                                             GAM00026
   80 WRITE(61,75)
      STOP                                                               C5107570
      END                                                                GAM00030
         FINIS
```



NOTE: TRM IS THE QUANTITY UNDER THE RADICAL IN EQUATIONS 12 AND 13.
VP IS THE VALUE OF EQUATION 13 WHEN THE POSITIVE VALUE IS USED.
VM IS THE VALUE OF EQUATION 13 WHEN THE NEGATIVE VALUE IS USED.
ENP IS THE VALUE OF EQUATION 12 WHEN THE POSITIVE VALUE IS USED.
ENM IS THE VALUE OF EQUATION 12 WHEN THE NEGATIVE VALUE IS USED.

COMPUTER PROGRAM FLOW CHART FOR ESTIMATING THE PARAMETERS OF A MIXED WEIBULL DISTRIBUTION

Load Data

First, two tables are read in starting in XI and FI

DG1 - $\Delta\gamma_1$

DG2 - $\Delta\gamma_2$

EG1 - Terminal γ_1

EG2 - Terminal γ_2

BGAM1 - Beginning γ_1

BGAM2 - Beginning γ_2

EN - Sample size. 2000 used for this program

ALP - α 0.8 used

NVI - Number of values in the tables

k - An integer $1 \rightarrow 4$

The computed μ_5' should approach m_5'

This can be accomplished by a series of runs varying $\Delta\gamma_1$, $\Delta\gamma_2$, γ_1, and γ_2

Legend

m_1' - PM1	θ_1 - TH1
m_2' - PM2	θ_2 - TH2
m_3' - PM3	
m_4' - PM4	
m_5' - PM5	
β_1 - B1	
β_2 - B2	
β_3 - B3	
ψ_1 - PSI1	
ψ_2 - PSI2	
ψ_3 - PSI3	
μ_5' - UP5	
N - EN	Sample Size
n - EN	Once the above N is used the location EN is no longer needed. It is used to store n.
v - V	

ENP - In equation (12), if the positive value of the radical is used.

ENM - If the negative value is used.

VP - In equation (13), if the positive value of the radical is used.

VM - If the negative value is used.

The following four combinations are possible, depending upon what number is read into location k.

v	n	k
+	+	1
+	-	2
-	+	3
-	-	4

Regardless of which combination is used, before the program is continued, the value of equation (13) is stored in V and the value of equation (12) is stored in EN.

REFERENCES

1. Cohen, A. Clifford, Jr.: Estimates in Mixtures of Two Poisson Distributions. Aero-Astrodynamics, Research and Development Research Review No. 1, NASA Technical Memorandum X-53189, October 1, 1964, pp.104-107.

2. Cohen, A. Clifford, Jr.: Estimation in Mixtures of Poisson and Mixtures of Exponential Distributions. NASA Technical Memorandum X-53245, George C. Marshall Space Flight Center, Huntsville, Alabama, April 9, 1965.

3. Cohen, A. Clifford, Jr. : Estimation in Mixtures of Two Normal Distributions. University of Georgia, Department of Statistics, Technical Report No. 13, 1965.

4. Pearson, Karl: Contributions to the mathematical theory of evolution. Philosophical Transactions of the Royal Society, London, 185, 1894, pp. 71-110.

5. Charlier, C. V. L.: Researches into the Theory of Probability. Meddelanden frau Lunds Astron. Observ., Sec. 2, Bd. 1, 1906.

6. Charlier, C. V. L.; and Wicksell, S. D.: On the Dissection of Frequency Functions. Arkiv för Matematik, Astronomi och Fysik, Bd. 18, No. 6, 1924.

7. Rider, Paul R.: Estimating the Parameters of Mixed Poisson, Binomial and Weibull Distributions by the Method of Moments. Bulletin de l'Institut International de Statistique, 1962, p. 39.

8. Cohen, A. Clifford, Jr.: On Some Conditions Under Which a Compound Normal Distribution is Unimodal. Technical Report No. 7, Contract DA-01-099 ORD-288, University of Georgia, 1953.

9. Cohen, A. Clifford, Jr.: Estimation in Mixtures of Discrete Distributions. Proceedings of the International Symposium on the Classical and Contagious Discrete Distributions, Montreal, Pergamon Press, 1963, pp. 373-378.

REFERENCES (Concluded)

10. Blischke, W. R.: Estimating the Parameters of Mixtures of Binomial Distributions. Journal of the Am. Stat. Association, 59, 1955, pp. 510-528.

11. Kendall, M. G.: The Advanced Theory of Statistics. Vols. I and II. Chas. Griffin and Co. Ltd. London, 1948.

12. Rao, C. Radhakrishna: Advanced Statistical Methods in Biometric Research, John Wiley and Sons, New York, 1952, pp. 300-304.

13. Rider, Paul R.: Estimating the Parameters of Mixed Poisson, Binomial, and Weibull Distributions by the Method of Moments. Bulletin de l'Institut International de Statistique, 39, Part 2, 1961, pp. 225-232.

14. Cohen, A. Clifford: Estimation in Mixtures of Discrete Distributions. University of Georgia, Institute of Statistics, Technical Report No. 16, 1963.

15. Whittaker, E. T; and Robinson, G.: The Calculus of Observations (second edition). Blackie and Son Limited, London, 1929.

16. Cohen, A. Clifford, Jr.: Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. University of Georgia, Institute of Statistics.

17. Menon, M. V.: Estimation of the Shape and Scale Parameters of the Weibull Distribution. Technometrics, Vol. 5, No. 2, 1963, pp. 175-182.

18. Kao, J. H. K.: Computer Methods for Estimating Weibull Parameters in Reliability Studies. Institute of Radio Engineers Transactions on Reliability and Quality Control,pp. 15-22, PGRQC-13, July 1958.

19. Cohen, A. Clifford, Jr.: Estimation in a Mixture of a Poisson with a Negative Binomial Distribution. University of Georgia, Institute of Statistics, Technical Report No. 15, 1965.

20. Kao, J. H. K.: A Graphical Estimation of Mixed Weibull Parameters in Life-Testing of Electron Tubes. Technometrics, Vol. I, 1959, pp. 389-407.

BIBLIOGRAPHY

Brown, George M.: On Sampling from Compound Populations. Annals of Mathematical Statistics, 4, 1933, pp. 288-342.

Hoel, Paul G.: Introduction to Mathematical Statistics. John Wiley and Sons, New York, 1963.

Whittaker and Watson: Modern Analysis. Cambridge University Press, Cambridge, 1958.

Kendall, M. G.; and Stuart, A.: The Advanced Theory of Statistics. Vol. 1, Hafner Publishing Co., New York, N. Y., 1958.

Kendall, M. G.; and Stuart, A.: The Advanced Theory of Statistics. Vol. 2, Hafner Publishing Co., New York, N. Y., 1958.

Kenney, J. F.; and Keeping, E. S.: Mathematics of Statistics. Part 2, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1939.

NASA-Langley, 1967 —— 19 M135

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546